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This teachers' guide was written to be used in conjunction with the student text, An Experimental Text in Transformational Geometry. The guide is intended to help teachers who have responsibility for teaching the topics Motions and Transformations in the Plane. Each section commences with a general discussion concerning the major ideas which are to be developed and understood by the students. In addition, situations and statements which could be difficult for students are identified. Finally, answers to questions and problems presented in the students' text are provided. [Not available in hard copy due to marginal legibility of original document]. (RP)

CCSM
Feasibility Study No. 43b

Teachers'
Guide

AN EXPERIMENTAL TEXT IN TRANSFORMATIONAL GEOMETRY

Educational Services Incorporated
Newton, Massachusetts

Prepared at the 1965
Mombasa Mathematics Workshop

Issued by
Cambridge Conference on School Mathematics
Educational Services Incorporated

CONTENTS

$$s_6 = \sqrt{2 \times 1^2 - 1\sqrt{4 \times 1^2 - 3}} \quad \text{and} \quad S_6 = \frac{2}{2\sqrt{3}}\left[\sqrt{4 + (2\sqrt{3})^2} - 2\right]$$

$$= 1 \qquad\qquad = \frac{2\sqrt{3}}{3}$$

Both of these agree with the previously known values for the sides of these polygons and this provides a check on our formulas. Applying the formulas again we find

$$s_{12} = \sqrt{2 - \sqrt{4 - 1}} \quad \text{and} \quad S_{12} = \frac{2}{\frac{2\sqrt{3}}{3}}\left[\sqrt{4 + \frac{4}{3}} - 2\right]$$

$$\approx \cdot 518 \qquad\qquad \approx \cdot 536$$

Also $s_{24} \approx \sqrt{2 - \sqrt{4 - \cdot 2680}}$ and $S_{24} \approx \frac{2}{\cdot 536}\left[\sqrt{4 + (\cdot 536)^2} - 2\right]$

$$\approx \cdot 260 \qquad\qquad \approx \cdot 264$$

From these values we find

$$p_3 = 3 \times \sqrt{3} \approx 5\cdot196 \qquad P_3 = 3 \times 2\sqrt{3} \approx 10\cdot392$$

$$p_6 = 6 \times 1 = 6\cdot00 \qquad P_6 = 6 \times \frac{2\sqrt{3}}{3} \approx 6\cdot928$$

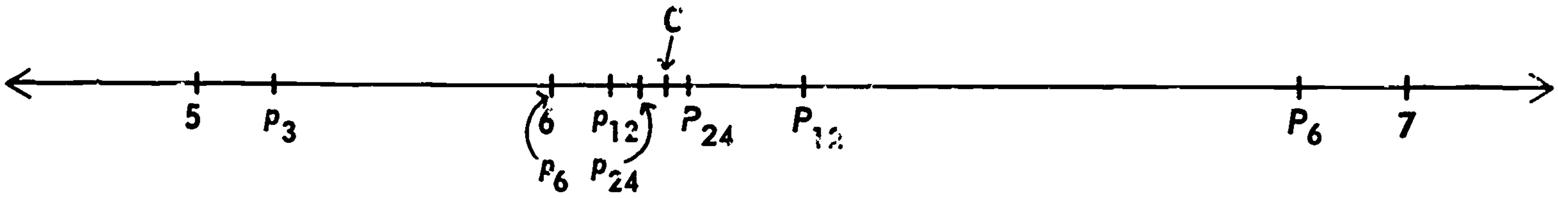
$$p_{12} \approx 12 \times (\cdot 518) \approx 6\cdot216 \qquad P_{12} \approx 12 \times \cdot 536 = 6.432$$

$$p_{24} \approx 24 \times (\cdot 260) \approx 6\cdot240 \qquad P_{24} \approx 24 \times \cdot 264 = 6\cdot336$$

We see that $P_3 - p_3 = 5\cdot18$; $P_6 - p_6 = 0\cdot92$
$P_{12} - p_{12} = 0\cdot216$; $P_{24} - p_{24} = 0\cdot096$. These differences are getting smaller and smaller as n increases. Since $p_n < C < P_n$ the average of p_n and P_n might be a good estimate of C. We note $\frac{p_3 + P_3}{2} = 7\cdot79$, $\frac{p_6 + P_6}{2} = 6\cdot46$, $\frac{p_{12} + P_{12}}{2} = 6\cdot32$ and $\frac{p_{24} + P_{24}}{2} = 6\cdot28$.

If we continued the computations for larger and larger n and assumed that r was exactly 1 so the results could be given to as many figures as we wanted, the differences $P_n - p_n$ would get closer and closer to 0. We already know that $6\cdot240 < C < 6\cdot336$ and we suspect that the average of these, i.e., $6\cdot28$ would be a pretty good value for C. As a matter of fact since $\pi = \frac{C}{2r}$ we would have $\pi \approx \frac{6\cdot28}{2 \times 1} = 3\cdot14$ and this is correct to three digits.

We now plot the values of p_i and P_i as we have found them so that we get a better picture of their relationship.



We could of course have started with $r = 1$, $s_4 = r\sqrt{2}$ and $S_4 = 2r$. In this way we would have obtained somewhat different approximations for C but by the time we had done the computations for $n = 8$, 16 and 32 we would again have obtained the value of C to be 6·28 to three digits or the value of π to be 3·14. There are much easier ways to compute π to many more places. These involve more advanced mathematics than your students have yet had but at least this relatively straightforward method gives us a reasonable approximation.

5-3 COMPUTING WITH π.

Many pupils will be tempted to give the circumference of a circle to many more digits than are justified. Do not ask for and do not allow pupils to give answers with greater accuracy than the initial data allow. The discussion in the text should make this clear. Usually answers in terms of π should be allowed unless otherwise specified.

Answers to
PROBLEMS 5-3

Student Text Pages 165-166

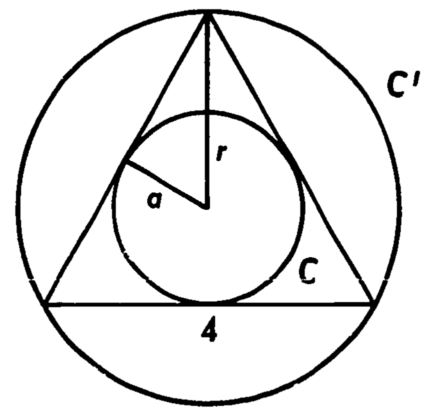
1.	a.	6π	d.	$2\sqrt{2}\,\pi$	g.	6π	j.	$\frac{6\pi}{7}$
	b.	10π	e.	$6\sqrt{5}\,\pi$	h.	π	k.	$\frac{4\pi}{5}$
	c.	34π	f.	$\frac{6\pi}{5}$	i.	$2\sqrt{2}\,\pi$	l.	14π

2.	a.	6	c.	$\frac{25}{2}$	e.	9	g.	$\frac{5}{\pi}$	i.	$\frac{11}{\pi}$
	b.	$\frac{17}{2}$	d.	$\frac{45}{4}$	f.	$\frac{\sqrt{2}}{2}$	h.	$\frac{18}{\pi}$	j.	$\frac{45}{2\pi}$

3. a. 20 c. 52·5 e. 45 g. 21 i. ·204
b. 38 d. 45·3 f. 19·7 h. 27·1 j. 204

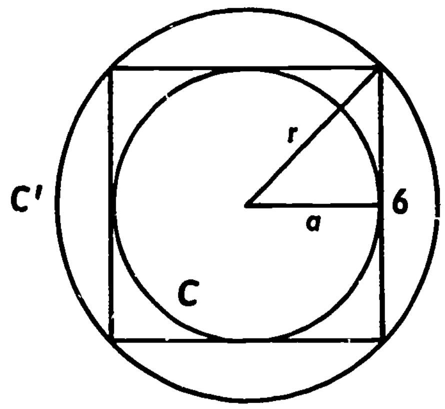
4. $s = 4$ $a = \frac{2}{3}\sqrt{3}$ $C = \frac{4\sqrt{3}\,\pi}{3}$

$r = \frac{4}{3}\sqrt{3}$ $C' = \frac{8\sqrt{3}\,\pi}{3}$



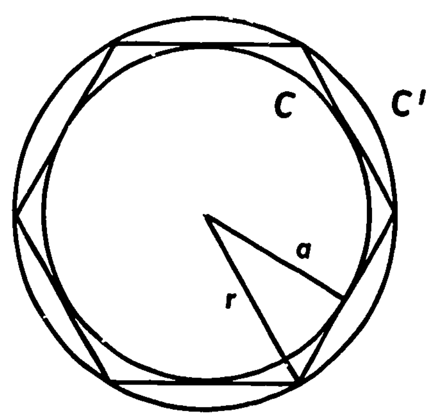
5. $s = 6$ $a = 3$ $C = 6\pi$

$r = 3\sqrt{2}$ $C' = 6\sqrt{2}\,\pi$



6. $s = 10$ $a = 5\sqrt{3}$ $C = 10\sqrt{3}\,\pi$

$r = 10$ $C' = 20\pi$



7. The polygon need not be regular.

Area $\triangle AOB = \frac{1}{2}r \times AB$

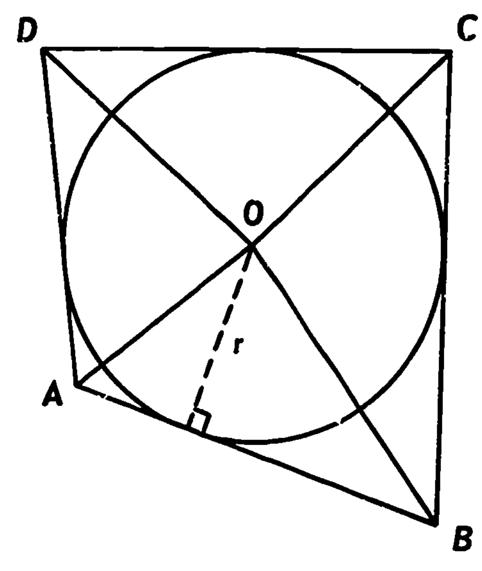
Area $\triangle BOC = \frac{1}{2}r \times BC$ etc.

Area of polygon = area $\triangle AOB$ + area $\triangle BOC + \ldots$

$= \frac{1}{2}r \times AB + \frac{1}{2}r \times BC + \ldots$

$= \frac{1}{2}r(AB + BC + \ldots + DA)$

$= \frac{1}{2}rp$



8. We know a circle may always be inscribed in a regular polygon. As in Problem 7,

Area $= \frac{1}{2}p \times$ (radius of inscribed circle)

$= \frac{1}{2}rp$

5-4 AREAS OF CIRCLES.

The area of a circular region, just as the circumference of a circle, is assumed to have a measure which may be approximated by the areas of the associated regular n-gons. Although we might carry out these successive approximations again it seems better simply to note that for the circumscribed polygons $A_n = \frac{1}{2} r P_n$ and that as n increases A_n gets close to A and P_n to C. The logical result is that $A = \frac{1}{2} r C$ which leads immediately to the familiar formula $A = \pi r^2$. The expression "area of a circle" is inaccurate since the circle has no area but nevertheless the meaning is clear and the phrase is much shorter than any more accurate one so we will use it freely.

Answers to PROBLEMS 5-4

Student Text Pages 167-170

1.	a.	25π	d.	2π	g.	$\frac{25\pi}{4}$	j.	64π
	b.	49π	e.	36π	h.	$\frac{49\pi}{4}$	k.	9π
	c.	$\frac{4\pi}{9}$	f.	144π	i.	$\frac{\pi}{9}$	l.	36π
2.	a.	8	d.	$\frac{3}{2}$	g.	$6\sqrt{2}$	j.	$\frac{5}{\pi}\sqrt{3\pi}$
	b.	11	e.	18	h.	$3\sqrt{5}$	k.	$\frac{10}{\pi}\sqrt{\pi}$
	c.	13	f.	$5\sqrt{2}$	i.	$2\sqrt{22}$	l.	1

3.	a.	$A = 75 = \pi r^2$	$r = \frac{5}{\pi}\sqrt{3\pi}$	$C = 2\pi r = 10\sqrt{3\pi}$
	b.	$\pi r^2 = 64\pi$	$r = 8$	$C = 16\pi$
	c.	$\pi r^2 = 8\pi$	$r = 2\sqrt{2}$	$C = 4\sqrt{2}\pi$
	d.	$\pi r^2 = 9\pi$	$r = 3$	$C = 6\pi$
	e.	$\pi r^2 = 1000\pi$	$r = 10\sqrt{10}$	$C = 20\sqrt{10}\,\pi$
	f.	$\pi r^2 = 125$	$r = \frac{5}{\pi}\sqrt{5\pi}$	$C = 10\sqrt{5\pi}$
	g.	$\pi r^2 = 324$	$r = \frac{18}{\pi}\sqrt{\pi}$	$C = 36\sqrt{\pi}$
	h.	$\pi r^2 = 425$	$r = \frac{5}{\pi}\sqrt{17\pi}$	$C = 10\sqrt{17\pi}$

4. Area of ring $= \pi \times 36 - \pi \times 16 = 20\pi$

Area of smaller circle is 16π $\therefore \dfrac{\text{area of ring}}{\text{area of smaller circle}} = \dfrac{20\pi}{16\pi} = \dfrac{5}{4}$.

5. Area of first circle $= \pi \cdot 2^2 = 4\pi$

Area of second circle $= 8\pi$

$\therefore \pi r^2 = 8\pi$ and the radius of the second circle $= 2\sqrt{2}$ inches

6. $\dfrac{r}{r'} = \dfrac{2}{3}$. If $r' = 3a$ then $r = 2a$

$A' = 9a^2\pi$, $A = 4a^2\pi$. So $\dfrac{A}{A'} = \dfrac{4}{9}$

7. $\dfrac{A}{A'} = \dfrac{50}{72} = \dfrac{25}{36}$. If $r = 15$ $A = \pi r^2 = \pi \times 225$

$A' = \pi r'^2$

$\therefore \dfrac{25}{36} = \dfrac{225\pi}{(r')^2 \times \pi} = \dfrac{225}{(r')^2}$ $\therefore \dfrac{5}{6} = \dfrac{15}{r'}$ $r' = 18$

The radius of the larger circle is 18 inches.

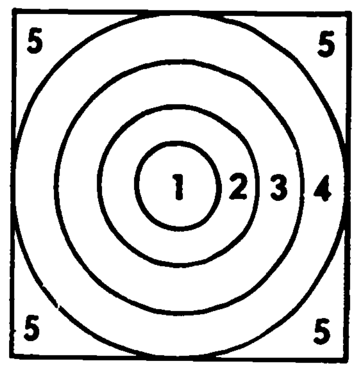
8. $A_1 = \dfrac{\pi}{4}$ $\approx \cdot 79$

$A_2 = \pi - \dfrac{\pi}{4} = \dfrac{3\pi}{4}$ $\approx 2\cdot 26$

$A_3 = \dfrac{9\pi}{4} - \pi = \dfrac{5\pi}{4}$ $\approx 3\cdot 93$

$A_4 = 4\pi - \dfrac{9\pi}{4} = \dfrac{7\pi}{4}$ $\approx 5\cdot 50$

$A_5 = 16 - 4\pi$ $\approx 3\cdot 44$

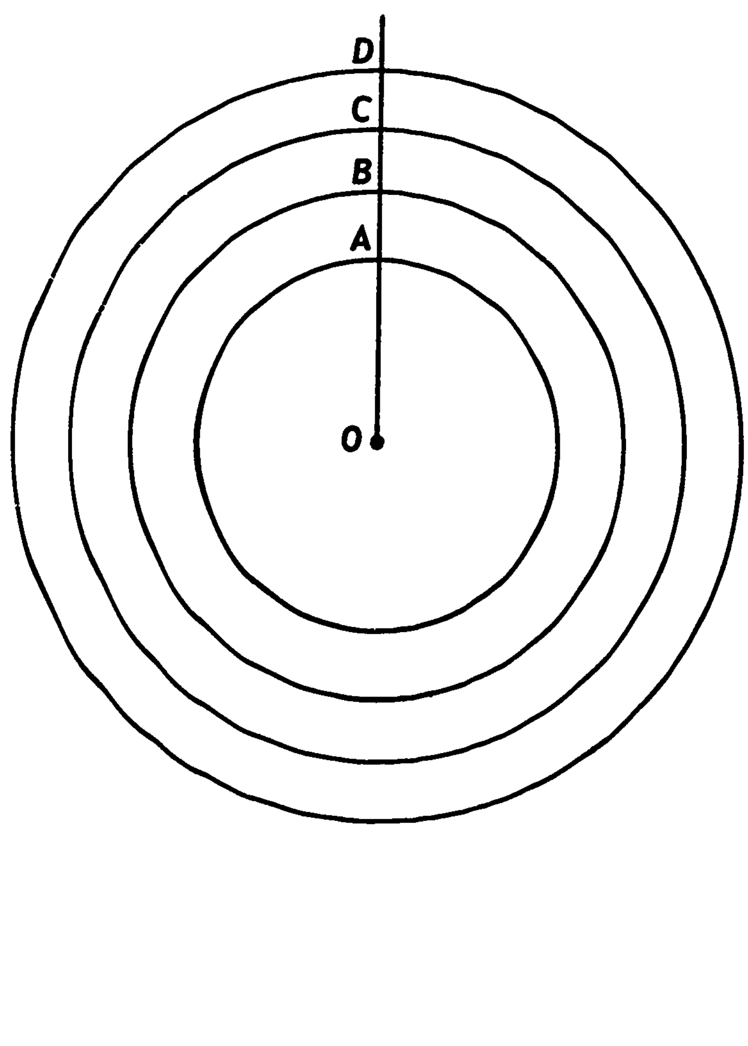


9. $\pi r_1^2 = \dfrac{1}{4} 4\pi = \pi$. $\therefore r_1 = 1 = OA$

$\pi r_2^2 = \dfrac{1}{2} 4\pi = 2\pi$. $\therefore r_2 = \sqrt{2} = OB$

$\pi r_3^2 = \dfrac{3}{4} \cdot 4\pi = 3\pi$ $\therefore r_3 = \sqrt{3} = OC$

Quite a surprising looking target isn't it?



10. $A = \pi r^2$
$A' = \pi r'^2$ $\quad \therefore \frac{A}{A'} = \frac{\pi r^2}{\pi r'^2} = \frac{r^2}{r'^2} = \left(\frac{r}{r'}\right)^2$

11. a. $\frac{1}{2}\pi r^2 = \frac{1}{2}100\pi = 50\pi$ since a diameter divides a circle in half.

b. $\frac{1}{4}\pi r^2 = \frac{1}{4}100\pi = 25\pi$. This is obviously half of the semicircular region.

c. $\frac{1}{2}2\pi r = 10\pi$

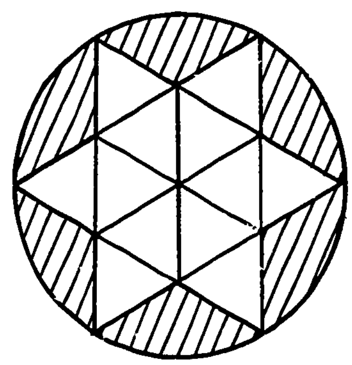
d. $\frac{1}{4} \cdot 2\pi r = 5\pi$

12. a. The two parts (shaded and unshaded) are the same shape so each must be half the circle; i.e., area = 50π.

b Perimeter = half of the large circle plus 2 halves of the small circle

$$= \frac{1}{2} \times 2\pi \times 10 + 2 \times \frac{1}{2} \times 2\pi \times 5$$
$$= 10\pi + 10\pi = 20\pi.$$

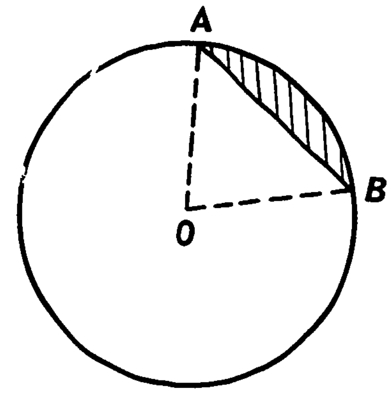
13. If $r = 6$, the side of the large equilateral triangles is $6\sqrt{3}$. The small triangles are also equilateral so each side = $2\sqrt{3}$. The unshaded part is made up of 12 such triangles. Its area is therefore $12 \times 3\sqrt{3} = 36\sqrt{3}$.



Area required is: $36\pi - 36\sqrt{3} =$
$36(\pi - \sqrt{3})$

5-5 LENGTHS OF ARCS, AREAS OF SECTORS AND SEGMENTS.

This section takes up lengths and areas of certain subsets of circles and circular regions. The fundamental technique is the use of the additive property of measure. We express the required point set as the union or intersection of point sets whose measures we already know and then add and subtract the corresponding measures as required. Thus the union of two semicircles of the same radius which have only their end points in common is the whole circle. The length of each semicircle is therefore half the length of the circle. Similarly if we have an arc of measure m in degrees, where m is an integer, the arc is the union of m arcs each of measure 1. The length of the arc of 1 degree measure is $\frac{1}{360}$ of the circle since the union of 360 such congruent arcs with only end points in common will be the circle. If m is not an integer we simply define the length of the arc as $\frac{m}{360} \times 2\pi r$ so that the same formula will always hold true. The same technique yields the formula for area of a sector.



For a segment bounded by $\overline{AB}$ and $\widehat{AB}$ we note that {Sector AOB } = the union of { ΔAOB} and {Segment AB }. Note the new use of the word segment. The "segment" $\overline{AB}$ is part of a line; "segment of a circle" is that part of a circular region cut off by a chord. Area of segment = area of sector - area of triangle. The difficulty here is that the area of ΔAOB is difficult to compute if only the radius of the circle and the measure of $\widehat{AB}$ are known. It can be done of course by use of the trigonometric formula area $\Delta AOB = \frac{1}{2} AO \times BO \times \sin A\hat{O}B$.

Answers to PROBLEMS 5-5

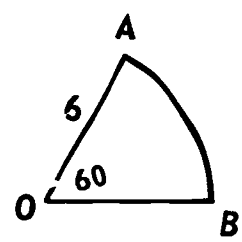
Student Text Pages 173-175

1. Perimeter $= OA +$ length of $\widehat{AB} + BO$

$$= 6 + \frac{1}{6} 2\pi \times 6 + 6$$

$$= 12 + 2\pi$$

Area $= \frac{1}{6}\pi \times 36 = 6\pi$

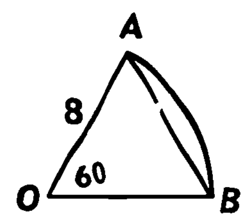


2. Perimeter $= m(\overline{AB}) + m(\widehat{AB}) = 8 + \frac{1}{6} 16\pi = 8 + \frac{8}{3}\pi$

Area $= m(\text{Sector } AOB) - m(\Delta AOB)$

$$= \frac{64\pi}{6} - 16\sqrt{3}$$

$$= \frac{32\pi}{3} - 16\sqrt{3}.$$



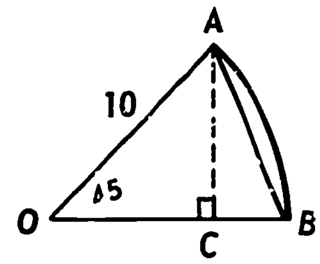
3. Area of Sector $AOB = \frac{45}{360} \times \pi \times 100 = \frac{25\pi}{2}$.

$m(\Delta AOB) = \frac{1}{2}\, OB \times AC$, but $AC = OC$

since $m(\widehat{OAC}) = 45$. $\therefore$ $AC = 5\sqrt{2}$

$m(\Delta AOB) = \frac{1}{2}\, 10 \times 5\sqrt{2} = 25\sqrt{2}$.

Area of segment $= \frac{25\pi}{2} - 25\sqrt{2}$



4. $\frac{m}{360} \times \pi \times 100 = 40\pi$

$\frac{10m}{36} = 40$

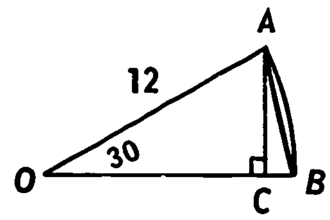
$m = 144$. (Degree measure of arc.)

length of arc $= \frac{144}{360} \times 2\pi \times 10 = \frac{2}{5} \times 20\pi = 8\pi$

5. Area $AOB = \frac{30}{360}\pi \times 12^2 = 12\pi$.

Area $\Delta\, AOB = \frac{1}{2}\, OB \times AC = \frac{1}{2}\, 12 \times 6 = 36$.

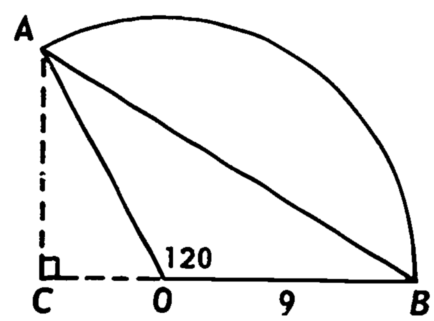
Area Segment $= 12\pi - 36$.



6. Area $AOB = \frac{120}{360}\, \pi \times 81 = 27\pi$.

Area $\Delta AOB = \frac{1}{2}\, OB \times AC = \frac{1}{2}\, 9 \times \frac{9}{2}\sqrt{3} = \frac{81\sqrt{3}}{4}$.

Area segment $= 27\pi - \frac{81\sqrt{3}}{4}$.



7. (3.) Major sector $= 100\pi - \frac{25\pi}{2} = \frac{175\pi}{2}$.

Major segment $= 100\pi - \left(\frac{25\pi}{2} - 25\sqrt{2}\right) = \frac{175\pi}{2} + 25\sqrt{2}$.

(5.) Major sector $= 144\pi - 12\pi = 132\pi$.

Major segment $= 144\pi - (12\pi - 36) = 132\pi + 36$.

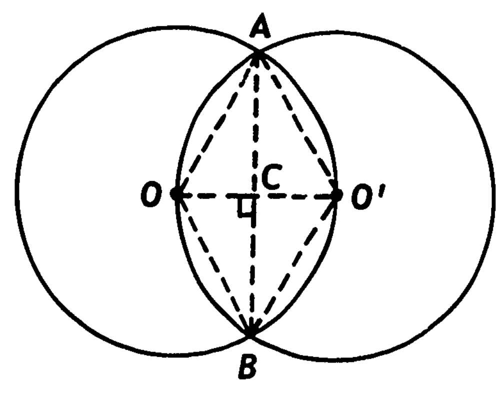
(6.) Major sector $= 81\pi - 27\pi = 54\pi$.

Major segment $= 81\pi - \left(27\pi - \frac{81\sqrt{3}}{4}\right) = 54\pi + \frac{81\sqrt{3}}{4}$.

8. Required area of intersection is sum of 2 segments.

$OA = OO' = AO' = 12$

$m(\widehat{AOB}) = 120$



Area segment $AO'B = \frac{120}{360} \times \pi 144 - \frac{1}{2} AB \times OC = 48\pi - \frac{1}{2} 12\sqrt{3} \times 6$

$= 48\pi - 36\sqrt{3}$

$\therefore$ Required area $= 96\pi - 72\sqrt{3}$.

Area of union = area of circle O + area of circle O' - area already found

$= 144\pi + 144\pi - (96\pi - 72\sqrt{3}) = 192\pi + 72\sqrt{3}$

9. $\frac{10}{360} \times \pi r^2 = \pi 15^2 - \pi 12^2$

$\frac{\pi r^2}{36} = 225\pi - 144\pi$

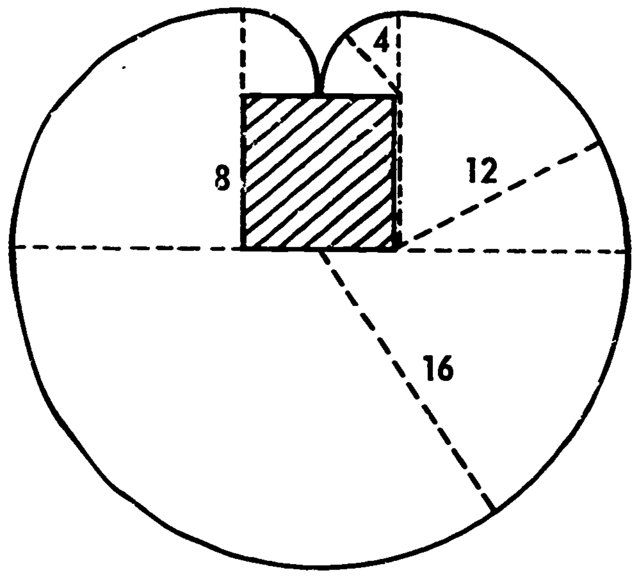
$r^2 = 36(81)$

$r = 6 \times 9 = 54.$

10. The area the goat can graze over is made up of one semicircular region of radius 16 feet, two quarter circle regions of radius 12 and two of radius 4. The total area is

$\frac{1}{2}\pi\ 16^2 + 2 \times \frac{1}{4}\pi\ 12^2 + 2 \times \frac{1}{4}\pi\ 4^2 =$

$128\pi + 72\pi + 8\pi = 208\pi$ sq. ft.

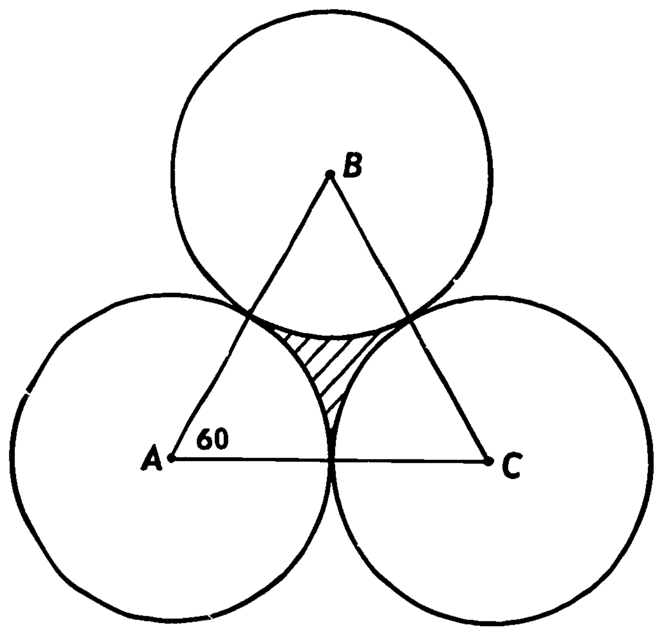


11. The required area is the difference between the areas of ΔABC and the three 60° sectors.

Area $\Delta ABC = \frac{1}{2} \times 4 \times 2\sqrt{3} = 4\sqrt{3}$

Area of each sector $= \frac{1}{6}\pi \times 4 = \frac{2\pi}{3}$

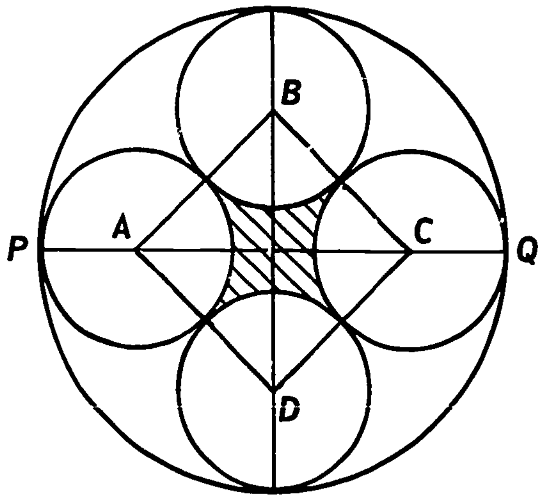
Answer $4\sqrt{3} - 2\pi$



12. The required area is the difference between the areas of the square $ABCD$ and the four 90° sectors. If the radius of each small circle is equal to x, $AB = 2x$.

Area = $4x^2 - \pi x^2$.

We still have to find x. $\overline{PQ}$ is a diameter. $PQ = 16$



$$AC^2 = AB^2 + BC^2 = (2x)^2 + (2x)^2 = 8x^2$$

$$AC = 2\sqrt{2}x, \ PA = x, \ CQ = x,$$

$$PQ = x + 2\sqrt{2}\,x + x = x(2\sqrt{2} + 2)$$

$$\therefore x(2\sqrt{2} + 2) = 16$$

$$x = \frac{8}{\sqrt{2} + 1} = 8(\sqrt{2} - 1)$$

$$x^2 = 64(2 - 2\sqrt{2} + 1)$$

$$= 64(3 - 2\sqrt{2})$$

$$\text{Area} = (4 - \pi)64(3 - 2\sqrt{2}) \approx 9{\cdot}3.$$

Since the area of the large circle ≈ 201, the shaded part is about 4·5% of the total.

5-6 RADIAN MEASURE.

This last section takes up the radian measure of an angle in anticipation of its use in trigonometry and, by those who take advanced mathematics, in calculus. The concept of using different units for angle measurement should not be too difficult for pupils who are used to using such different units as inches or centimetres for measuring segments. However they will need a lot of practice in converting from one type of unit to the other.

Answers to PROBLEMS 5-6

Student Text Pages 177-178

1. a. $\frac{\pi}{6}$ d. $\frac{7\pi}{180}$ g. $\frac{3\pi}{4}$ j. $\frac{\pi}{5}$

   b. $\frac{2\pi}{9}$ e. $\frac{\pi}{8}$ h. $\frac{7\pi}{8}$ k. $\frac{2\pi}{5}$

   c. $\frac{\pi}{12}$ f. $\frac{5\pi}{12}$ i. $\frac{179\pi}{180}$ l. $\frac{4\pi}{5}$

2. a. 90 d. $\frac{432}{\pi}$ g. $\frac{108}{\pi}$ j. 100

b. 36 e. 50 h. $\frac{234}{\pi}$ k. 157·5

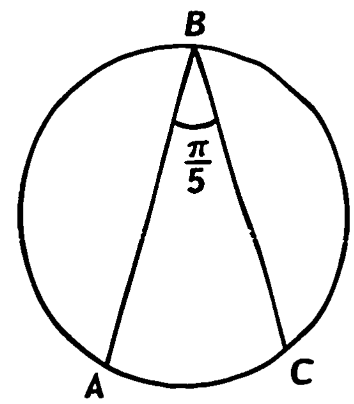
c. $\frac{270}{\pi}$ f. 120 i. $\frac{180}{7}$ l. $\frac{180}{\pi}$

3. a. π d. $\frac{2\pi}{3}$ g. ·1225 j. 2

b. $\frac{8\pi}{3}$ e. 6 h. 4

c. π f. 9·1 i. 9

4. $m(A\hat{B}C) = \frac{\pi}{5}$

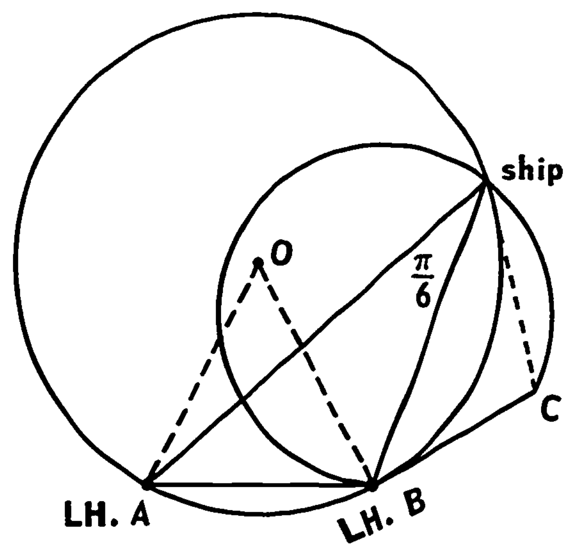


$\therefore m(\widehat{AC}) = \frac{2\pi}{5}$ or

$\widehat{AC}$ is one fifth of the circle. The length of $\widehat{ABC}$ is $\frac{4}{5}$ of the circumference

$\therefore$ Length of $\widehat{ABC} = \frac{4}{5}\ 24\pi = \frac{96\pi}{5}$.

5. The angle at the ship is $\frac{\pi}{6}$ radians.



Suppose we could draw the circle determined by the ship and the two lighthouses.

Then $m(A\hat{O}B) = 2m(A\hat{S}B)$

$m(A\hat{O}B) = \frac{\pi}{3}$

This means that $\widehat{AB}$ is an arc whose degree measure is 60 and ΔAOB is an equilateral triangle. We can then find the position of O by drawing two circles with centres at A and B and each with radius 10 miles. O will be the intersection of these circles on the seaward side of $\overline{AB}$. Draw the circle with centre O and radius 10. The ship is somewhere on the major arc of this circle since at any point on this arc the measure of the angle subtended by $\widehat{AB}$ will be equal to $\frac{\pi}{6}$.

If a third lighthouse were visible at some point C a similar construction would put the ship on a circle of which $\overline{BC}$ was a chord. The intersection of these two circles would be the position of the ship.

Chapter 6
MOTIONS AND TRANSFORMATIONS

INTRODUCTION.

Introductory Comments. The ideas presented in this chapter have been well known in advanced mathematics for a long time. It is only in the past few years, however, that they have begun to appear in secondary school texts. As you will see, these ideas are simple, natural, and directly related to other major areas of mathematics, such as algebra. Many educators and mathematicians believe that, in years to come, these ideas will become more and more important in secondary mathematics.

Because these ideas are probably new to you as well as to your pupils, they are presented slowly and with care in the Student Text. *It is vitally important that you give the entire Student Text and the entire Teacher's Guide (for this chapter) a careful reading before you begin to teach this material.* In this way you will get an over-all view of the direction and main ideas of the material, and you will see how very simple and elementary these ideas are, once they are properly understood. In your careful preliminary reading, pay special attention to Sections 6-1 to 6-6. Also, be sure to look at all the problems and at the discussion of answers given in this Guide. The problems are meant to be especially helpful for obtaining an easy and correct understanding.

We now give a brief general summary of the over-all content of Chapter 6. We follow this with a brief summary by sections. We then give some more general comments. We then discuss various ways of omitting material to shorten the program. We then give several suggested programs. Finally, we end the Introduction with a discussion of how to plan your time in presenting Chapter 6 to your pupils. After the Introduction, we take up the Student Text, section by section, giving answers to problems and giving other general comments on the material.

You will probably find that you will understand the summary below better after you have worked through the entire Text and Guide.

Summary. This chapter presents the concept of *rigid motion* in the plane. Various kinds of rigid motion are considered, certain mathematical facts about rigid motions are obtained, and a number of applications are described. A rigid motion is a *mapping* (that is to say, *function*) from the plane onto the plane such that the distance between the images (under the mapping) of any two points is the same as the distance between the two points. Special kinds of rigid motion include direct motions, reversing motions, translations, rotations, and reflections. One of the chief mathematical facts presented is that every rigid motion can be viewed either as a translation, a rotation, a reflection, or a combination of reflection and translation. (Which of these a motion is will depend upon the particular motion.) This fact and others lead to a variety of useful applications in geometry. Also, certain algebraic notations and ideas can be used in working with rigid motions.

Summary by Sections. Sections 6-1 and 6-2 use exercises with tracing paper to help prepare and build the pupil's thinking. Questions of congruence and of "moving" geometrical figures in the plane are the immediate subject matter. In Section 6-3, the concept of rigid motion is introduced and several basic mathematical facts about rigid motions are given. Section 6-4 gives two illustrations of the way in which the concept of rigid motion can help our thinking in solving geometrical problems. In Section 6-5, the basic kinds of rigid motion are described and illustrated. In Section 6-6, we use the idea of rigid motion to give a general definition of congruence for geometrical figures in the plane. Sections 6-7, 6-8, and 6-9 give further information and examples about translations, rotations, and reflections. In Section 6-10, we consider ways that different rigid motions can be combined, one after the other, to give new rigid motions, and we see the usefulness of algebraic ideas in working with such combined motions. In Section 6-10, we also present some of the main theoretical facts, such as, for example, the fact that every direct motion is either a translation or a rotation. Finally, in Section 6-11, we give several further applications of rigid motions to solving geometrical problems.

General Comments. (1) As you read through the Text and Guide, you see that our work on rigid motions in Chapter 6 is closely related to the work on algebra in *Secondary Four*. Both in the algebra and in the geometry of rigid motions, the concept of *function* is fundamental. In algebra, we have functions from real numbers to real numbers. In geometry, a rigid motion is a special kind of function from points in the plane to points in the plane. In Section 6-3 of the Student Text, this connection between the pupil's work in geometry and algebra is emphasized, and various notations used in the pupil's work are also used for rigid motions. In studying Section 6-3, you should read again the basic material on *functions* in the algebra Student Text and Teacher's Guide.

(2) A main reason for studying rigid motions is that they have a close connection with more advanced work in algebra as well. This is shown a little bit in the work on coordinate axes in Sections 6-7, 6-8, and 6-9, and it is shown in the work in Section 6-10 on combinations of motions. It will become much clearer in *Secondary Five* algebra where the pupil will be introduced to matrices and to the concepts of linear algebra which are fundamental for so much of modern mathematics and its applications. The pupil will then find that his study of rigid motions will have provided him with a valuable preparation for this later work in algebra.

(3) The value of this study of rigid motions comes, in large part, from the simplicity and naturalness of the geometric ideas. The work in Chapter 6 is intended to show this. If your pupils find Chapter 6 easy, do not be dismayed. This only means that you are doing a successful job in teaching the material to them.

Material that can be omitted. The Student Text is arranged so that, depending on time available, on the background and enthusiasm of your pupils, and on your personal preferences, certain material can be omitted. This material, which we list below, is in part theoretical. While of value, it is not essential. Even if you omit all of the material below, many important points and ideas will remain, and the student will still have a sound and worthwhile program upon which to build later mathematical work. Do not hesitate to drop any of this material that does not suit your class, your time, or your preferences.

Material can be omitted, as you choose, according to the following outline, which proceeds section by section.

6-1. All of this section should be used.

6-2. The last part, after Problems 6-2B, can be omitted or treated lightly. This part, on "using ruler and compasses," will be of more interest to pupils who have already had ruler and compasses constructions up to the construction of a line through a given point parallel to a given line. Note the important final sentence of Section 6-2 (just before Problems 6-2C). Rigid motion ideas are very useful for solving problems about ruler and compasses constructions, but knowledge of ruler and compasses constructions is not necessary for understanding the theory of rigid motions.

6-3. All of this material should be used. Note, however, that certain proofs (of properties (2)-(6) from property (1)) have been placed in an Appendix. These proofs are intended as "footnotes" to the text. Later work does not depend on them. Except for pupils with a special interest in them, the proofs in this Appendix should be omitted.

6-4. The first example should be used, but the second can be omitted or treated lightly, especially if pupils have not studied ruler and compasses constructions. (See comment above on possible omission of material from Section 6-2.)

6-5. All of this section should be used.

If you are pressed for time, you can stop with 6-5 and omit the rest of Chapter 6. Your class will still have obtained much of value. If you go further, you can take one of the following alternatives.

(a) Do Sections 6-6 through 6-9.
(b) Do Sections 6-6 through 6-10.
(c) Do Sections 6-6 through 6-11

In each of these alternatives, there are partial omissions possible in the sections listed. We now describe these.

6-6. The "Note on isometric mappings" at the end of the section can be omitted. It is not difficult, however, and should be kept if possible.

6-7. The proof following Definition 6-2 can be either omitted or treated briefly. The material on coordinate axes can be omitted.

6-8. The proof following Definition 6-3 can be either omitted or treated briefly. The material on symmetry and coordinate axes can be omitted.

6-9. The proof following Definition 6-4 can be either omitted or treated briefly. The material on symmetry and coordinate axes can be omitted.

6-10. This section can be done even if all the above omissions in Sections 6-7, 6-8, and 6-9 have been made. Section 6-10 is necessary for Section 6-11. The material under "A basic theorem" (after Problems 6-10D) should be included, except that the construction after the statement of Theorem 6-1 can be omitted.

6-11. Either or both of Examples 3 and 4 can be omitted.

Appendix. These proofs can be omitted as noted above in the comment on Section 6-3.

Suggested program. What program you choose, and how much you omit, will depend on four things: (i) your own preferences as to emphasis; (ii) how much time you have available; (iii) the interest and previous preparation of your pupils; and (iv) the examination syllabus for which your pupils are preparing. The minimum program noted above (Sections 6-1 through 6-5 with some omissions) provides the pupil with much of value for later work in mathematics. At the present writing (1965) an examination syllabus has not been prepared for Secondary 4. When such a syllabus is drawn up, we would expect it to cover material in Sections 6-7, 6-8, and 6-9 as well, and possibly some material from Section 6-10. The following comments on time may help you in planning a program for your class.

Planning your time. We assume five class meetings per week. We estimate that the sections will take the following amounts of time. (You may find that the material goes somewhat faster than this.)

Sections 6-1 and 6-2 together will take about a week. There is not much mathematical content, but the exercises need to be covered with some care. These sections will take less than a week if the ruler and compasses part of Section 6-2 is omitted.

Sections 6-3 and 6-4 together will take about a week. Again, the total amount of material is not great, but there are new ideas and new terminology that need to be treated with care. These sections will take less that a week if the second illustration in Section 6-4 is omitted.

Sections 6-5 and 6-6 together will take about a week, and less if the last part of Section 6-6 is omitted.

(Thus the minimum program mentioned above, of Sections 6-1 through 6-5 with omissions, can be covered in two to two and a half weeks.)

Sections 6-7, 6-8, and 6-9 together can be covered in two weeks. With omissions, they can be covered in a week and a half.

Section 6-10 will take about a week.

(Thus a minimum program leading through Section 6-10 would take from four to five weeks.)

Section 6-11 will take a little less than a week.

(Thus a maximum program through Section 6-11 and including all material could take as much as seven weeks.)

Special materials needed. In addition to the usual materials for geometry, each pupil should be supplied with about twenty sheets of tracing paper. Any paper thin enough to make tracings from a pencilled drawing will serve.

6-1 FIGURES WITH THE SAME SIZE AND SHAPE.

General discussion. Read this entire section with care. You will need two class meetings to cover this section. In the first meeting you can discuss the section with the class, go over the examples in the text, and carry out the class activity. In the second meeting you can go over the problems with the class. These problems, while easy, are of central importance for giving the pupil a full understanding. You can also invent additional problems as we point out below.

The main idea of this section is simple. In previous work two triangles were defined to be "congruent" if they had corresponding sides congruent, and corresponding angles congruent. The pupil proved as a theorem (SSS) that if corresponding sides are congruent, then corresponding angles are congruent also. The notion of congruence was important because it made precise the idea of two triangles having "the same size and shape". Can we state the idea of "same size and shape" in a more general way that will apply to other figures besides triangles? Section 6-1 answers this question in a simple and obvious way. Two figures have the same size and shape if a tracing of one figure can be made to coincide with the other. This answer is not a purely mathematical one, but it leads us, in later sections, to a variety of interesting mathematical ideas. Sections 6-3 and 6-6 will show how the answer can be made purely mathematical. If pupils ask you for a more precise answer at this stage, probably the best reply is to tell them to imagine a "perfect" tracing, and then to say to them that two figures have the "same size and shape" if a perfect tracing of one figure can be made perfectly to coincide with the other.

As the text makes clear, there is one aspect of our answer that the pupil may find unnatural at first. This is the fact that we allow our tracing to be turned over. (And thus we say that in Figure 7, for example, A and B have the same size and shape.) The pupil may not feel that this agrees with his own private idea of "same shape". In any case, he must understand that this is the way the words "same shape" are to be used in the present chapter.

Warning. There are two places where your pupils may be confused by the words we use. (1) We use the word "figure" sometimes to mean a single geometrical figure such as a triangle, a square, or a circle, and sometimes we use the word "figure" to mean an entire drawing in the text such as "Figure 7" or "Figure 11." Once the pupil is aware that "figure" is used in these two distinct ways, he will have no trouble. Which of the two meanings is intended will always be clear from surrounding statements. Thus, in the title of Section 6-2 ("Moving a figure in the plane"), it is the first of the above meanings that is intended. (2) In the paragraph following Figure 7, we speak of "turning over" the tracing paper. By this we mean turning the paper entirely over so that the side of the paper which was previously underneath and facing the figure being traced is now on top and facing you. (Some pupils may make the mistake at first of thinking that "turning over" means sliding the paper through a rotation of 180° while always keeping the same side of the paper in contact with the figure being traced.)

Answers to questions raised in text.

The triangles in Figure 1 do have the same size and shape. So do the triangles in Figure 2, the circles in Figure 3, and the circles in Figure 4. The ruler and compasses construction to find the centre of a circle is to draw two non-parallel chords and find their perpendicular bisectors. The intersection of these bisectors is the centre of the circle. You may wish to omit the exercise with Figure 4 (since it will require drawing on the pages of the text) and to put a blackboard exercise in its place.

The two figures in Figure 5 have the same size and shape, as we can see by sliding a tracing of one over the other. In the Class Activity, the figures in (a) and (c) have the same size and shape, but the figures in (b) do not.

Answers to PROBLEMS 6-1

Student Text Pages 183-185

1. (a) yes; (b) yes; (c) yes (in this case, the tracing paper must be turned over.)

2. (a) 1; (b) 2; (c) 6; (d) 4; (e) 4.

Comment. The purpose of these problems is not to teach a special technique of using tracing paper, but rather to prepare the student for the mathematical idea of a *rigid motion*. Once the student sees how the tracing paper is used, he may find that he is able to answer certain questions (like Problem 2 above) by imagining the correct motion of the tracing paper without actually carrying it out physically. This should be encouraged. Problems similar to Problem 2 can be easily made up by you for use in class. For example, the following figures can be used.

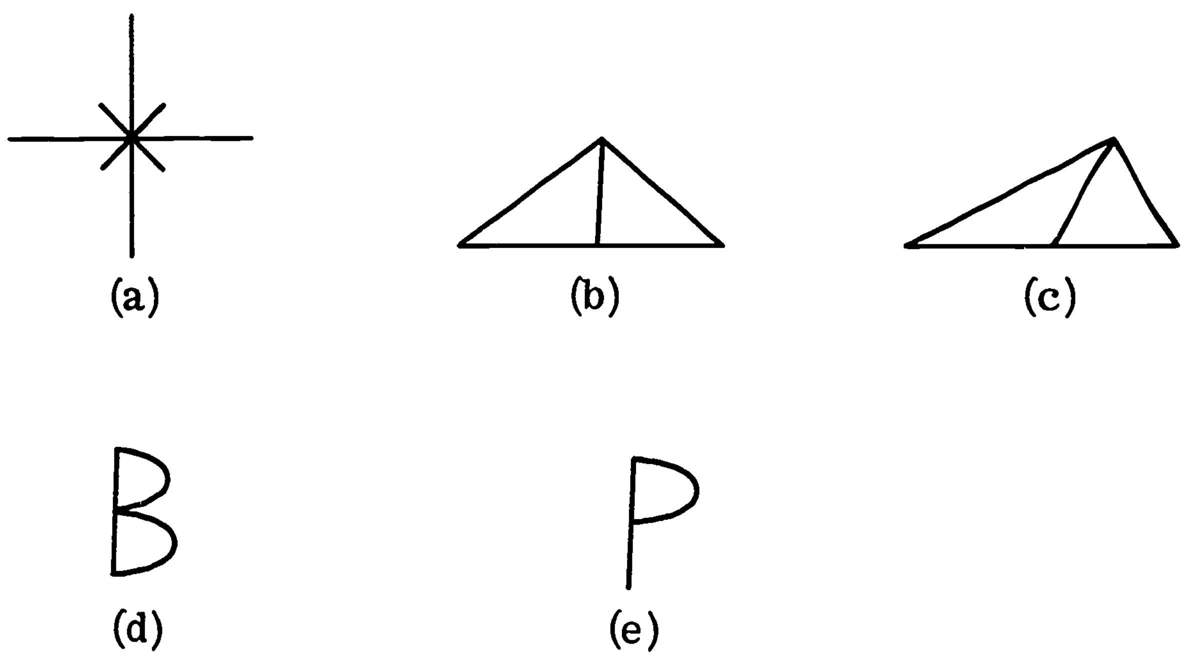


Fig. G-1.

Here the answers are: (a) 8; (b) 2; (c) 1; (d) 2; (e) 1. (Letters of the alphabet make a good source of problems.)

6-2 MOVING A FIGURE IN THE PLANE.

General discussion. The material in Section 6-2 falls into three parts. The first part (up to Problems 6-2A) shows how we can use tracing paper to "move" a figure from one place to another in the plane. The second part (up to Problems 6-2B) shows that, along with the figure being moved, any such "movement" by tracing paper carries every other point in the plane to a new position. The third part (up to Problems 6-2C) shows that ruler and compasses can often be used without tracing paper to get the same result, in moving a figure or a point, that we would get with tracing paper.

The first two parts will take one class meeting each. The third part, if you decide to use it, will take one or possibly two class meetings. As with Section 6-2, the problems are especially important for the pupil to get a full understanding and you should spend time in class discussing them. A good class activity in the second part is to have your pupils use tracing paper to check the statements made about Figures 16, 17, and 18 in the text.

As we remarked in the Introduction, the third part can be omitted. If you omit the third part, Problems 4 and 5 (and possibly 6) in Problems 6-2C should still be given to your pupils.

Warning. Two difficulties can arise in connection with the basic steps described at the beginning of Section 6-2. (1) How is the student to make a copy of Figure 10? One answer is to have the student make a tracing of Figure 10 on tracing paper. If he does this, however, he may become confused between his two pieces of tracing paper in carrying out the steps described in the text. A better way is to have him make the copy on a sheet of paper which is somewhat thicker than tracing paper, but which is still thin enough to trace onto from the Figure in the book. A third and still better way is to have a copy of Figure 10 already prepared for his use. (These remarks also apply to later Figures and Problems.) (2) How can we be sure that the student is following the basic steps correctly? The main thing here is to have him see

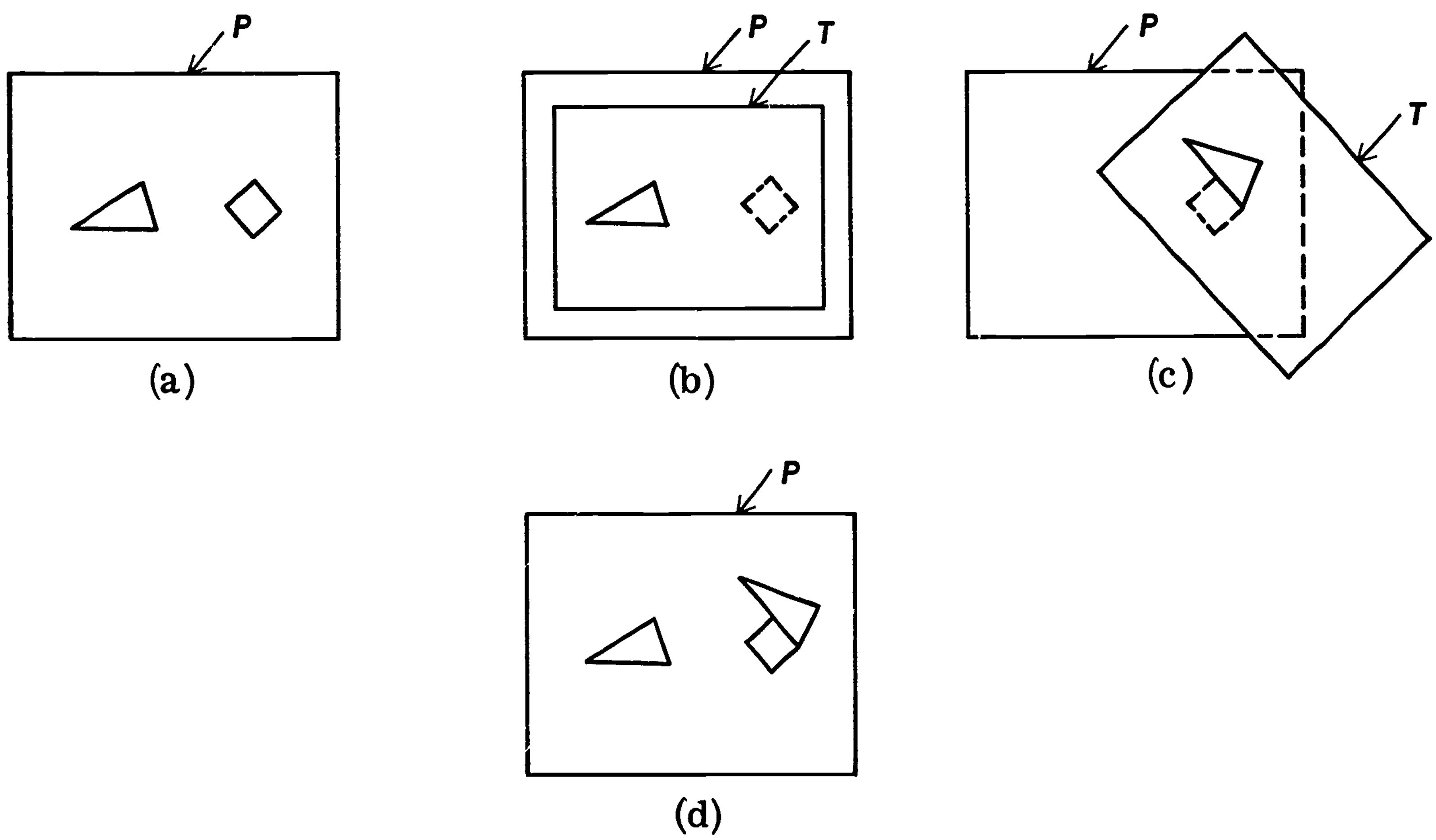


Fig. G-2.

that P is the piece of paper with the original copy of Figure 10 and T is the piece of tracing paper which he is using to move the triangle to a new position on P. The successive steps can be pictured as in Figure G-2. In (a) we have the copy P of Figure 10. In (b) we have placed the tracing paper on top and made a tracing of the triangle. In (c) we have moved the tracing paper. (d) gives the result on P after we have pricked through the vertices of the triangle from T onto P and then drawn its sides on P.

Be sure to emphasize to your pupils that for certain motions (Problem 3 below, for example) they will have to turn paper T over before pricking through onto P.

Answers to
PROBLEMS 6-2A

Student Text Pages 186-187

1. Figure 44 (in the Student Text) gives one solution.
2. Figure 45 (in the Student Text) gives one solution.
3. Figure 46 (in the Student Text) gives the solution.
4. The four answers to Problem 1 are indicated in Figure G-3. Note that for two of these answers the tracing paper must be turned over. The four answers to Problem 2 are indicated in Figure G-4.

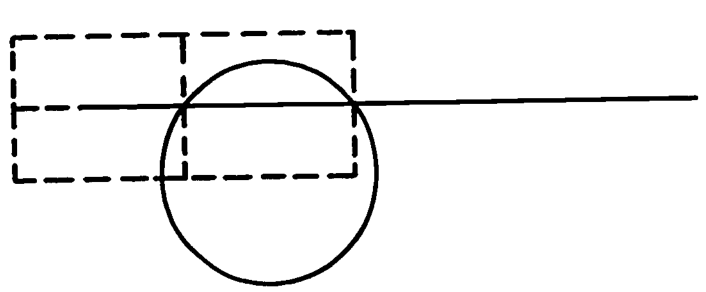


Fig. G-3.

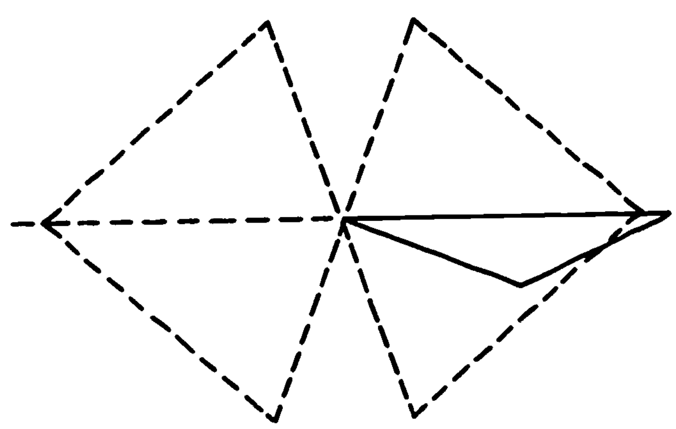


Fig. G-4.

1. In this movement every point moves the same distance (such a movement will be called a *translation* in Section 6-5), so that there is no fixed point. Figure G-5 shows the new positions of P_1, P_2, and P_3 as P_1', P_2' and P_3'.

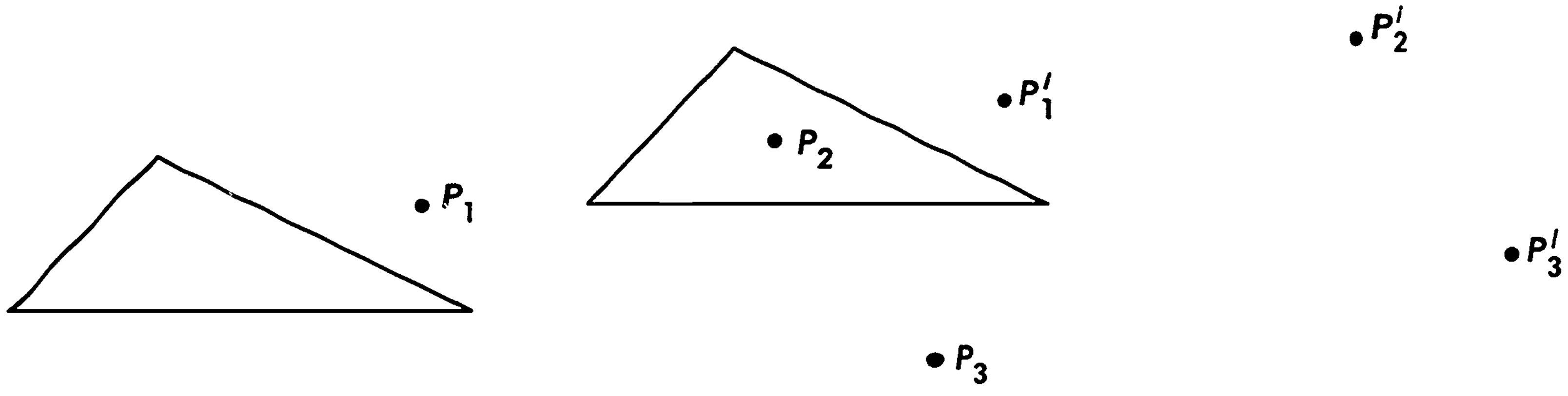


Fig. G-5.

2. Points P_1, P_2 and P_3 are moved as shown in Figure G-6. The circle D is invariant under this movement as can be checked with tracing paper. The centre of the circle is a fixed point of the movement. (This movement is an example of what will be called, in Section 6-5, a *rotation.*)

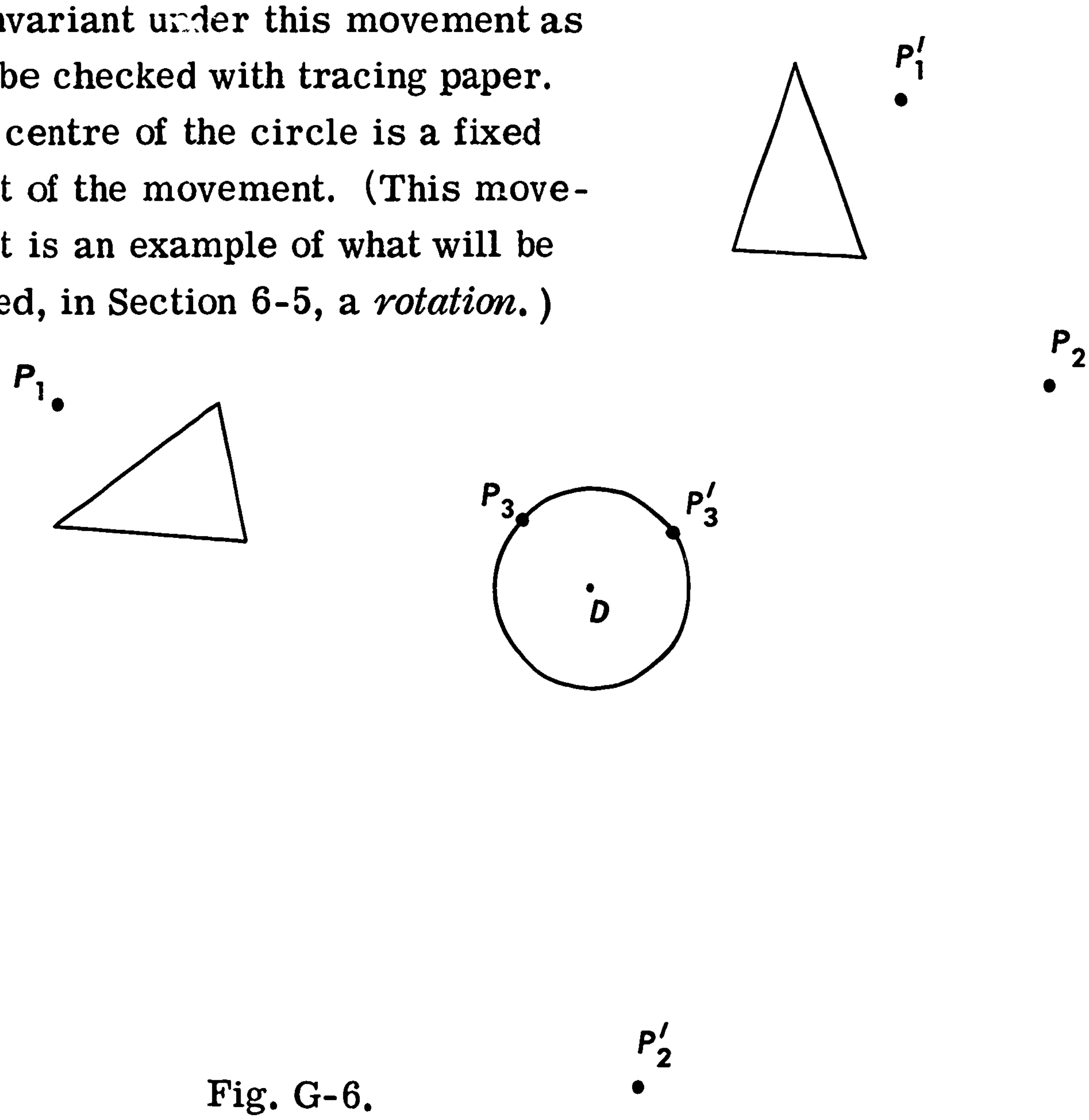


Fig. G-6.

3. This carries L_1 and L_2 onto themselves, and L_3 to the position L_3' in Figure G-7. (This movement is an example of what will be called, in Section 6-5, a *reflection.*)

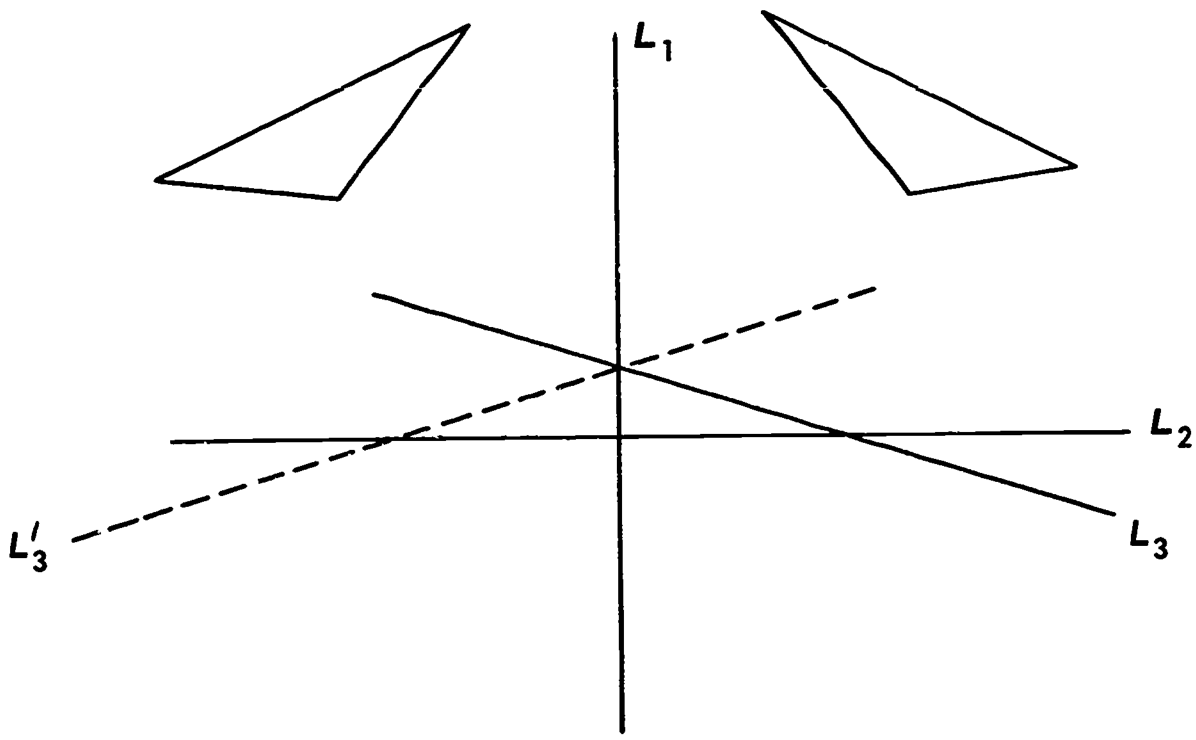


Fig. G-7.

4. (a) The fixed points are the points of line L_1.
(b) L_1 is invariant and any line perpendicular to L_1 (such as L_2) is invariant. (Note, however, that the only fixed point on L_2 is its point of intersection with L_1.)

5. There are six motions, and the new positions of P are as in Figure G-8.

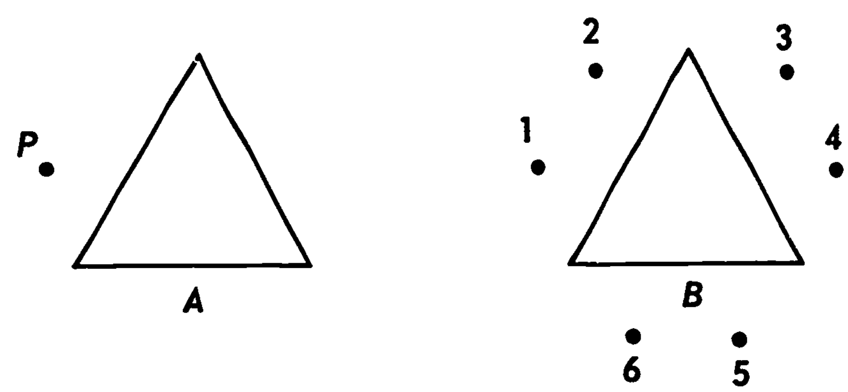


Fig. G-8.

1. Find B' on CD so that $CB' = CB$. Then draw an arc of radius AB about B' and an arc of radius AC about C. Take an intersection of these arcs as A'.

2. Find B' between D and F so that $B'F = BA$. Then take an arc of radius AC about F and an arc of radius BC about B'. Their intersection inside the given circle yields the third vertex C'.

3. Construct the desired rectangle on L by constructing a perpendicular to L at E, laying off $A'E = AD$ on L and $C'E = CD$ on the perpendicular, and then getting B' as the intersection of two appropriate arcs with centers at A' and C'.

4. Let D' be the new position of D. To locate D', we take an arc of radius AD about A' and an arc of radius CD about C'. These arcs will intersect at two points. D' will be the intersection which lies on the opposite side of $\overline{A'C'}$ from B' (since D lies on the opposite side of $\overline{AC}$ from B.) The new positions of E and F can be found similarly. Note that we could have equally well used arcs around A' and B' or around C' and B' to locate D'.

5. (a) The points on a circle of radius AP about A'.
   (b) The two points given by the two intersections of an arc of radius AP about A' with an arc of radius BP about B'.
   (c) The single point given when we take the two intersections as in (b) and then choose that intersection which lies on the opposite side of $\overline{A'B'}$ from C'.

6. There are a variety of acceptable answers. We give two examples here.

   Example 1. "If we are given three points not all on the same straight line, and if we know where a movement carries each of those three points, then we can find where the movement carries any fourth point."

   Example 2. "Let A and A' be any two points, then there are many different movements which carry A to A'. Let A, B, A', and B' be four points such that A is different from B and $AB = A'B'$.

Then there are exactly two movements which carry A to A' and B to B'. Let A, B, and C be any three points not on a straight line, and let A', B', and C' be any three points such that $AB = A'B'$, $BC = B'C'$, and $AC = A'C'$. Then there is exactly one movement which carries A to A', B to B', and C to C'."

6-3 RIGID MOTIONS.

General discussion. This section has two main parts. In the first part, we see that any "movement" of tracing paper gives us a *function* or *mapping* from the plane onto itself. (For the purpose of seeing this, we imagine that our tracing paper is infinitely large and covers the whole plane.) Any such function, given by a tracing paper movement, is called a *rigid motion.* We here use the word "function" in exactly the same way that the word "function" is used and discussed in *Secondary Four* algebra. Ask your pupils to go back and revise their work on functions in algebra. The *domain* of a rigid motion is the whole plane, and the *range* of a rigid motion is the whole plane. In the first part of Section 6-3, we give notations and terms that will be used in the study of rigid motions. The pupil will have met some of these already in his work on algebra. If T is a rigid motion and P is a point, and if T carries P to Q, then we use the functional notation of algebra and write "$T(P) = Q$." We call Q the *image* of P under the rigid motion T. P is sometimes called a "pre-image" or "inverse image" of Q under the rigid motion T. The result of not moving the tracing paper at all is also taken to be a rigid motion and is called the "identity motion." It is like the identity function in algebra.

The second part of the section begins with the list of six properties that all rigid motions have. It presents these properties to the pupil and introduces several additional new words. The most important property is the first property, which states that given any rigid motion T and given any two points P and Q, the distance between $T(P)$ and $T(Q)$ is the same as the distance between P and Q. This property of rigid motions is called the *isometric* property. (The word "isometric" comes from the Greek: *iso* + *metron* where *iso* = same and *metron* = measure.) Later, in the part of

Section 6-6 called *Note on Isometric Mappings*, we shall show a fundamental fact about this property: not only does every tracing paper motion have this property, but, also, every function from the plane into the plane with this property can be obtained from an appropriate tracing paper motion. We state this fundamental fact for the pupil in Definition 6-1.

The Student Text has been written so that your pupils can approach the subject of rigid motions in either of two quite different ways. (1) The pupil can continue to think of a rigid motion as something got by moving tracing paper. In the second part of Section 6-3, he then simply observes, as a fact about tracing paper, that every rigid motion must have each of the six properties listed. For him, the definition of *rigid motion* is like the definition of *polynomial function* in algebra—namely, it is a function which can be actually carried out in a certain way (by a certain movement of tracing paper, which is like, in algebra, calculating values of a certain polynomial.) This approach to rigid motions has the advantage that it is easier for the pupil to follow and understand. You will probably wish to have most of your pupils take this approach. The Text is written so that pupils can follow this approach to the end of the Chapter. This approach has the disadvantage, however, that it depends on an essentially non-mathematical idea: the idea of tracing paper. (2) The pupil can *define* rigid motions to be those functions from the plane into the plane which happen to be isometric. This makes the concept of rigid motion purely mathematical, but at the expense of requiring that the pupil accept and use a more abstract idea. Probably, most of your pupils will not be ready for this approach. The Text is written, however, so that pupils who wish to, can also follow this approach to the end of the chapter. If this approach is taken, then the remaining five properties of rigid motions become mathematical facts to be proved. Geometric proofs of these facts are given in the Appendix at the end of the chapter. If your pupils follow the first approach, you may still wish to point out to them that the second approach exists, even though you are not using it. (Tell them that Definition 6-1, using the isometric property, is just a way of making the idea of tracing paper mathematically precise.) As the Text remarks, the more abstract approach is necessary when we study rigid motions in three-dimensional (solid) geometry.

Although the ideas are simple (especially if you use the first approach mentioned above), you will want to be sure that your students can use these ideas carefully and correctly. This means that you should spend three or four days on Section 6-3: one day for each of the two parts of the section, and one or two days to go over problems and examples.

Warning. The word "movement" may be confusing to some of your pupils. They may think that "rigid motion" means a particular *path* of movement. This is not true. A rigid motion is determined by the initial and final positions of the tracing paper. If the tracing paper moves through two different paths to get to the same final position, then it is the same rigid motion, whichever path is used.

Comments on motion geometry. In the study of plane geometry, other functions besides the rigid motions are sometimes used, and additional terminology is used. A function with properties (3) and (4) (and hence also (2) and (6), see Problem 12) is called a *transformation,* or, sometimes, an *affine transformation.* A function with these properties and property (5) as well is sometimes called a *homothetic transformation,* or a *similarity transformation,* or, simply, a *motion.* Functions with property (1) as well, that is to say, rigid motions, are sometimes also called *Euclidean transformations.* In recent years, these various functions have been used more and more in presenting geometry to secondary pupils. Problem 7 gives an example of a transformation which is not homothetic, and Problem 9 gives an example of a homothetic transformation which is not a rigid motion.

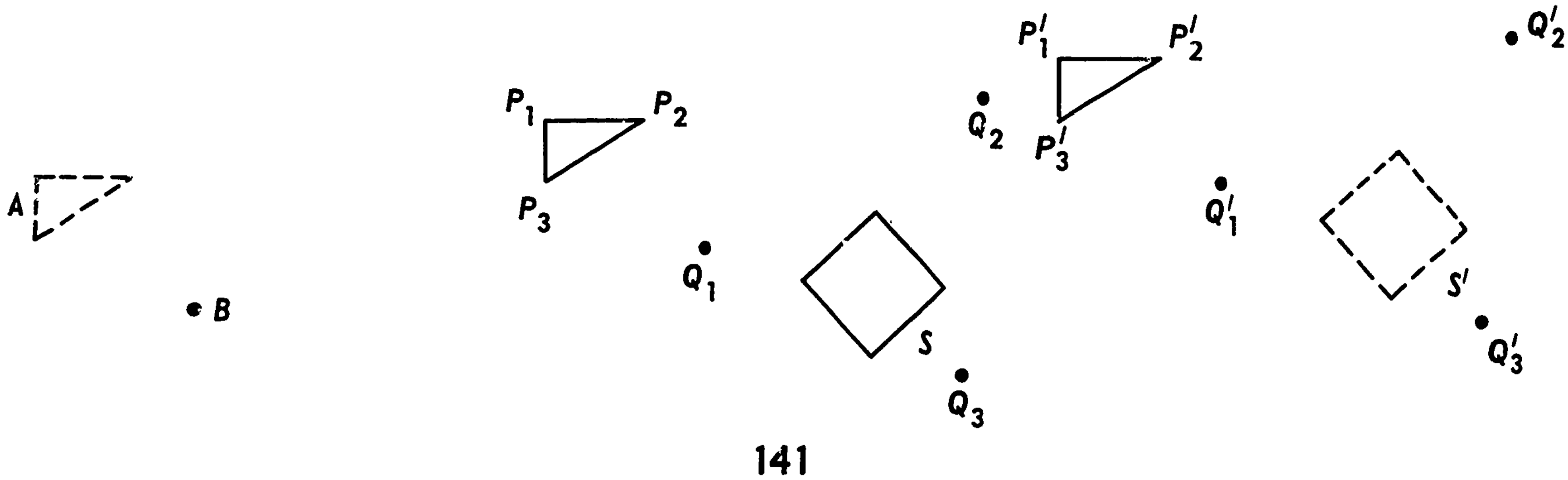
Answers to
PROBLEMS 6-3.

Student Text Pages 200-204

1. In Figure G-9, Q_1', Q_2' and Q_3' are the images of Q_1, Q_2 and Q_3 and S' is the image of S.

2. In Figure G-9, B has Q_1 as its image, and triangle A has $\Delta P_1P_2P_3$ as its image. (This rigid motion is an example of what will later be called a *translation.*)

Fig. G-9.



3. The image of $\Delta P_1'P_2'P_3'$ is $\Delta P_1P_2P_3$, hence $\Delta P_1'P_2'P_3'$ is, itself, the triangle which has $\Delta P_1P_2P_3$ as its image. (This rigid motion is an example of what will later be called a *reflection.*)

4. In Figure G-10, $\Delta P_1''P_2''P_3''$ is the image of $\Delta P_1'F_2'P_3'$, and triangle A has $\Delta P_1P_2P_3$ as its image. (This rigid motion is an example of what will, in Section 6-10, be called a *glide reflection.*)

5. (a) No. A shift of the tracing paper, such as in Problem 1 above, has no fixed point. Every point gets moved the same distance.

(b) Yes. In the identity motion, for example, every line is identical with its image. Are there examples other than the identity motion? Yes, the shift of Problem 1 is such a motion.

(c) No. A rotation of the tracing paper through 90° will, for example, make every line perpendicular to its own image. (See Problem 2 of Problems 6-2B.)

(d) No. See (c) above.

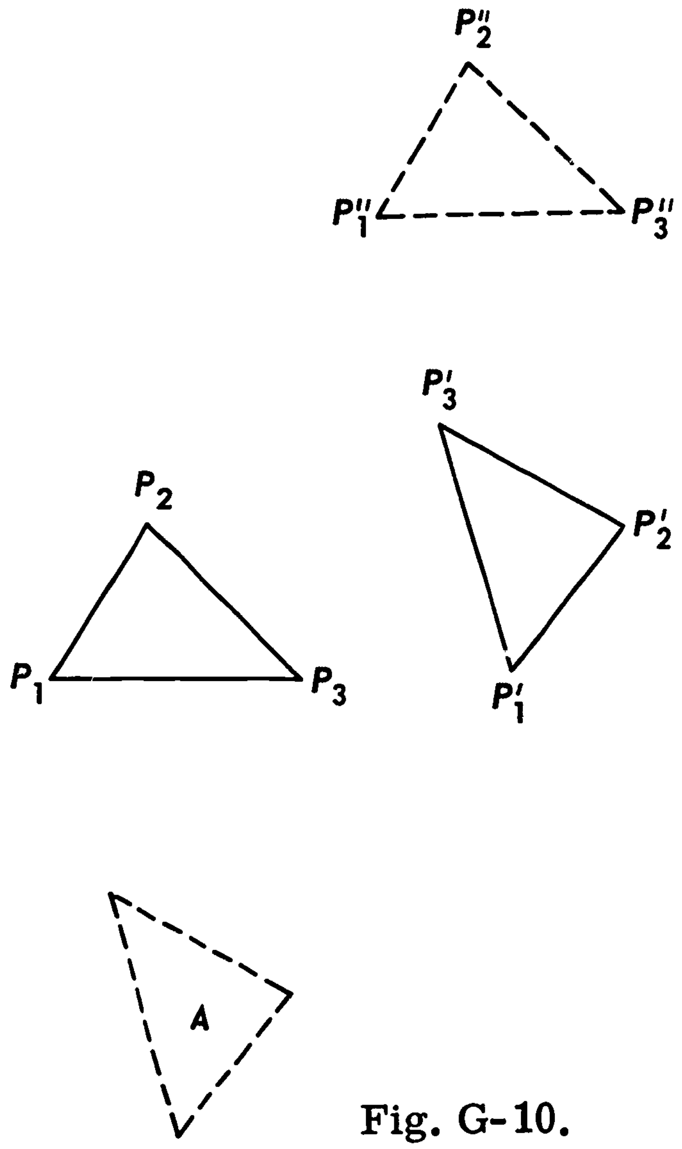


Fig. G-10.

(e) Yes. In Problem 1, the line $\overleftrightarrow{P_1P_1'}$, for example, is its own image. (If students ask for a "proof" of this, tell them to wait until Section 6-7.)

(f) No. Assume that a motion maps some triangle onto itself. Either it maps all three vertices onto themselves (and we have the identity motion) or it maps some one vertex onto another and we have a triangle that must be isosceles or equilateral. In the former case, every point is a fixed point. In the latter case, the intersection of the angle bisectors must be a fixed point. (Since every angle bisector gets mapped onto an angle bisector, a point that lies on all three angle bisectors must lie on all three images of angle bisectors.)

Hence any motion which maps a triangle onto itself must have a fixed point.

6. We use the notation $\begin{pmatrix}ABC\\DEF\end{pmatrix}$ to stand for that rigid motion which takes A to D, B to E, and C to F. Similarly we use $\begin{pmatrix}ABC\\EFD\end{pmatrix}$ to stand for that rigid motion which takes A to E, B to F, and C to D. Similarly for the other four possibilities. We consider each of the six possibilities in turn.

(i) $\begin{pmatrix}ABC\\DEF\end{pmatrix}$. This motion shifts every point the same distance to the right, and there are no fixed points.

(ii) $\begin{pmatrix}ABC\\EFD\end{pmatrix}$. This motion is given by a rotation of the tracing paper about a fixed point P_1 in Figure G-11. It has P_1 as its one and only fixed point.

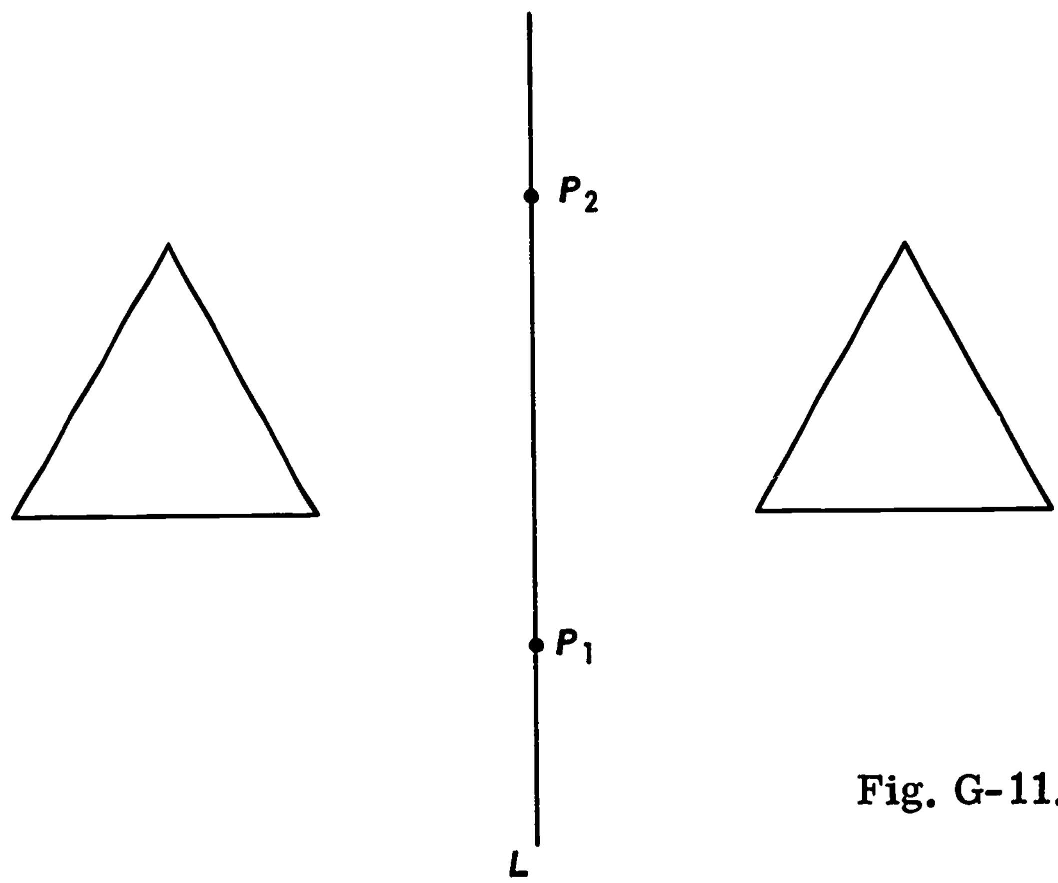


Fig. G-11.

(iii) $\begin{pmatrix}ABC\\FDE\end{pmatrix}$. This motion is given by a rotation about point P_2. It has P_2 as its one and only fixed point.

(iv) $\begin{pmatrix}ABC\\FED\end{pmatrix}$. This motion requires that the tracing paper be turned over and put back down in such a way that it has the line L as its set of fixed points.

(v) $\begin{pmatrix}ABC\\EDF\end{pmatrix}$. This motion requires that every line parallel to $\overleftrightarrow{AB}$ be moved perpendicular to itself. Hence there are no fixed points.

(vi) $\begin{pmatrix}ABC\\DFE\end{pmatrix}$. This motion requires that every line parallel to $\overleftrightarrow{BC}$ be moved perpendicular to itself. Hence there are no fixed points.

7. This problem is easier than the pupil may expect. Let P be any point not on L and let Q be the foot of the perpendicular from P to L. Then $P'Q' = 2(PQ)$. Since $PQ \neq O$, $PQ \neq P'Q'$. Hence the mapping T is not isometric. Hence it is not a rigid motion.

8. This construction amounts to turning the tracing paper over and placing it back down so that every point of L falls on its original position. The "Hint" suggests a more formal, geometrical proof. This proof is given with Figure 80 in Section 6-9 of the Student Text.

9. The proof that T is not a rigid motion is similar to that in Problem 7. Take P different from O. Then $OP \neq O$ and $O'P' = OP' = 2(OP)$.

To show that T is a transformation, we must show that it has properties (3) and (4).

Any point P is the image of the point halfway between P and O. This shows that (3) holds.

To show (4), we must show that the image of every line is a line. To do this, let L_1 be any given line. If L_1 contains O, then L_1 is immediately seen to be its own image. If L_1 does not contain O, let P and Q be any two distinct points on L_1. Let L_2 be the line through the points $T(P)$ and $T(Q)$. Let M be a line through O intersecting L_1 and L_2 at R and R'. Then, by a similar triangles argument, $OR = RR'$. Hence $R' = T(R)$. This shows that every point on L_1 is carried to a point on L_2. and that every point on L_2 comes from some point on L_1. Hence L_2 is the image of L_1. See Figure G-12. To show that T has property (5), it is enough to show that the image of any triangle A is a triangle similar to A. To show this, it is enough (because of a basic SSS theorem on similar triangles) to show that the image of any segment is a segment that is twice as long. To show this latter fact, consider the segment $\overline{PQ}$ and its image in Figure G-12.

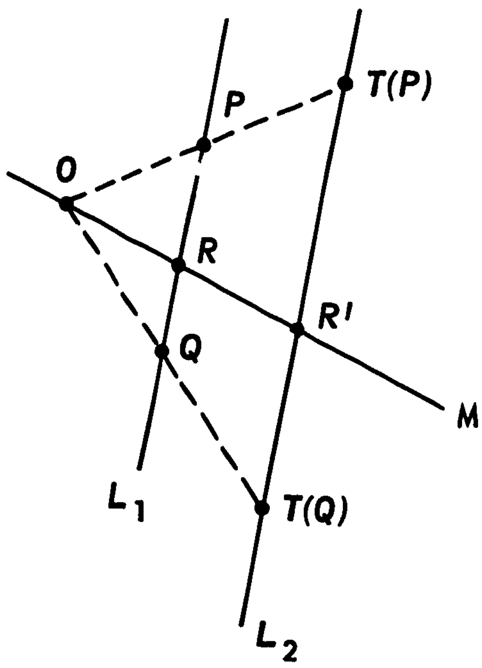


Fig. G-12.

10. The figure on the right shows that property (5) fails, since $P'\widehat{Q'}R'$ is contained in, and hence smaller than, $P\widehat{Q}R$. (Here Q' is the same point as Q.)

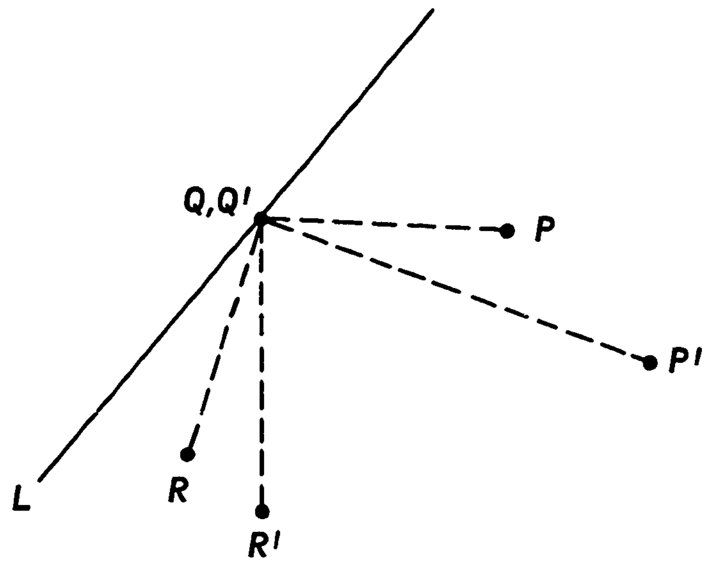


Fig. G-13.

11. Any point P is the image of the point halfway between P and L. This shows that property (3) holds. To show (4), we must show that the image of every line is a line. The proof is similar to that for Problem 9. Let L_1 be any given line. If L_1 is identical with L, then clearly the image of L_1 is L_1 itself. If L_1 is parallel to L, then clearly the image of L_1 is a line parallel to L and twice as far from L. If L_1 intersects L at some point Q, let P be any point on L_1 different from Q. Find $T(P)$ and let L_2 be the line determined by Q and $T(P)$. Let M be any line perpendicular to L, and let R and R' be its intersections with L_1 and L_2. Let P_1 be the foot of the perpendicular from P to L and let R_1 be the intersection of M with L. By the construction, we have $P_1P =$ distance from P to $T(P)$. Hence by similar triangles, $R_1R = RR'$. Hence $R' = T(R)$. This shows that every point on L_1 is carried to a point on L_2, and that every point on L_2 comes from some point on L_1. See Figure G-14.

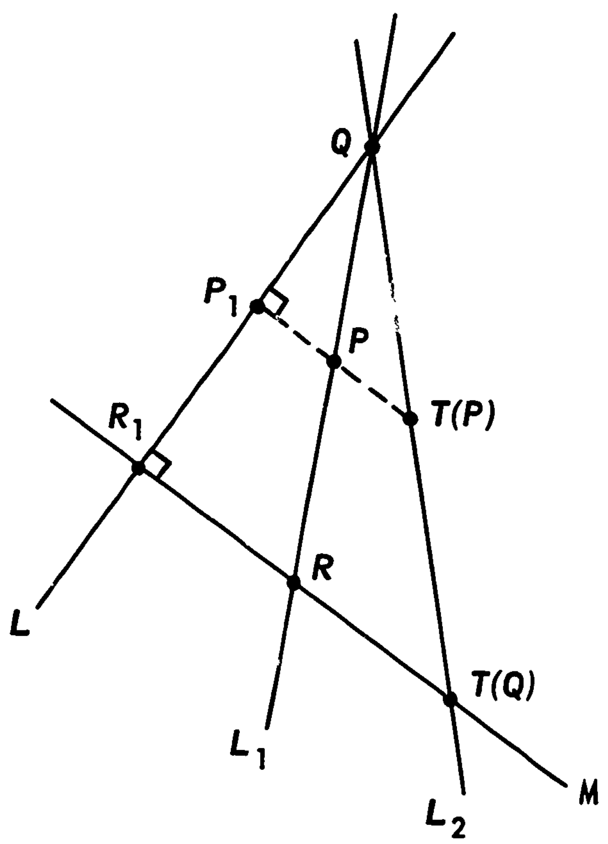


Fig. G-14.

12. We first show (2) and then deduce (6) from (2). Let T be any transformation. (That is to say, T has properties (3) and (4).) We show (2) by assuming that (2) is false and getting, from this, a contradiction.

If (2) is false, there are distinct points P_1 and P_2, and a point Q, such that $T(P_1) = T(P_2) = Q$. Let L be $\overleftrightarrow{P_1P_2}$. Then by (4), the image of L is a straight line M. Let Q' be a point not on M. Then, by (3), Q' has a pre-image under T. Let P' be a pre-image of Q'. Then P' cannot lie on L. Let L_1 be $\overleftrightarrow{P_1P'}$ and let L_2 be $\overleftrightarrow{P_2P'}$. Let M' be $\overleftrightarrow{QQ'}$. Then M' must be the image of both L_1 and L_2. Let M'' be any line parallel to M'. Take distinct points Q_1 and Q_2 on M''. Let P_3 be a pre-image of Q_1 and P_4 be a pre-image of Q_2. P_3 and P_4 must be distinct, since Q_1 and Q_2 are distinct. Let L'' be $\overleftrightarrow{P_3P_4}$. Then L'' must intersect L_1 or L_2. Also L'' has M'' as its image. As we saw on the preceeding page, L_1 and L_2 have M' as their common image. Hence the point of intersection between L'' and L_1 or L_2 must map to a point of intersection between M' and M''. But M' and M'' are parallel. This is a contradiction. We have hence shown (2). See Figure G-15.

It is easy now to go and show (6). See the last proof in the Appendix.

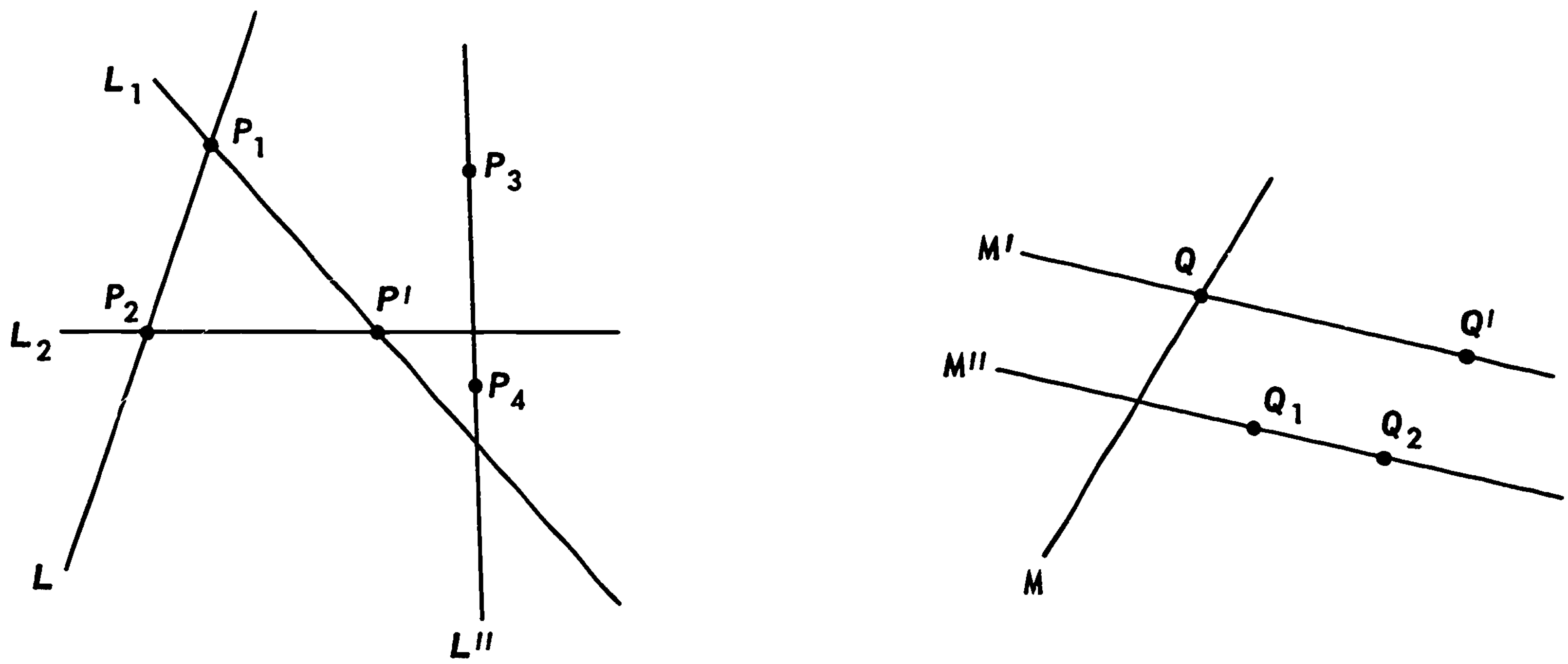


Fig. G-15.

Note. Even though the proof uses only elementary ideas, Problem 12 above is perhaps the most difficult of all the problems in Chapter 6. If any of your pupils get a correct solution to this problem, it is likely that they have mathematical abilities of a high order, and that they will be able to pursue a university course in advanced mathematics with distinction.

A somewhat easier problem that you can also give to your students is the following: show that if a function from the plane into the plane has properties (2) and (4), then it must have property (3). An outline of the solution is as follows. Let Q be any point. We wish to show that Q is the image of some point. Take any two lines L_1 and L_2. Let M_1 and M_2 be the images of L_1 and L_2. Take any line through Q that intersects M_1 and M_2, and let Q_1 and Q_2 be the points of intersection. Let P_1 and P_2 be the pre-images of Q_1 and Q_2. Then $\overleftrightarrow{P_1P_2}$ must have $\overleftrightarrow{Q_1Q_2}$ as its image, and hence the point Q since it lies on $\overleftrightarrow{Q_1Q_2}$ must have a pre-image.

6-4 USING MOTIONS TO SOLVE PROBLEMS

General discussion. This section can be treated rather briefly. It is intended to show the pupil some of the ways in which the idea of rigid motion can be useful. It is separate from the work in other sections, and later sections do not depend on it. You should allow one or two class meetings for it (only one meeting if the second example is treated lightly or omitted.) Example 1 is a practical problem where the use of a rigid motion gives a quick and neat answer. Example 2 is a geometrical construction which appears difficult until we find that, by thinking about the correct motions, we can organize our attack on it in a rather simple way. The final ruler and compasses solution which we get does not speak of motions, even though we use motions to discover it.

The main point of this section is not the particular examples given, and it is not to give special ways of solving problems. The main point is that the general idea of motion can often be surprisingly helpful in both practical and mathematical thinking.

Answer to Challenge Problem. Let A be the centre of C_1. Construct a line parallel to L through A. (See the construction for Figure 28 in Section 6-2.) Find a point B on this line such that $AB = PQ$. Draw a circle with centre B and radius same as C_1. Let P_1 and P_2 be the points of intersection of this circle with C_2. Draw lines through P_1 and P_2 parallel to L. Let Q_1 and Q_2 be the intersections of these lines with C_1 as shown in the figure below. Then $\overline{P_1Q_1}$ and $\overline{P_2Q_2}$ are the two possible segments.

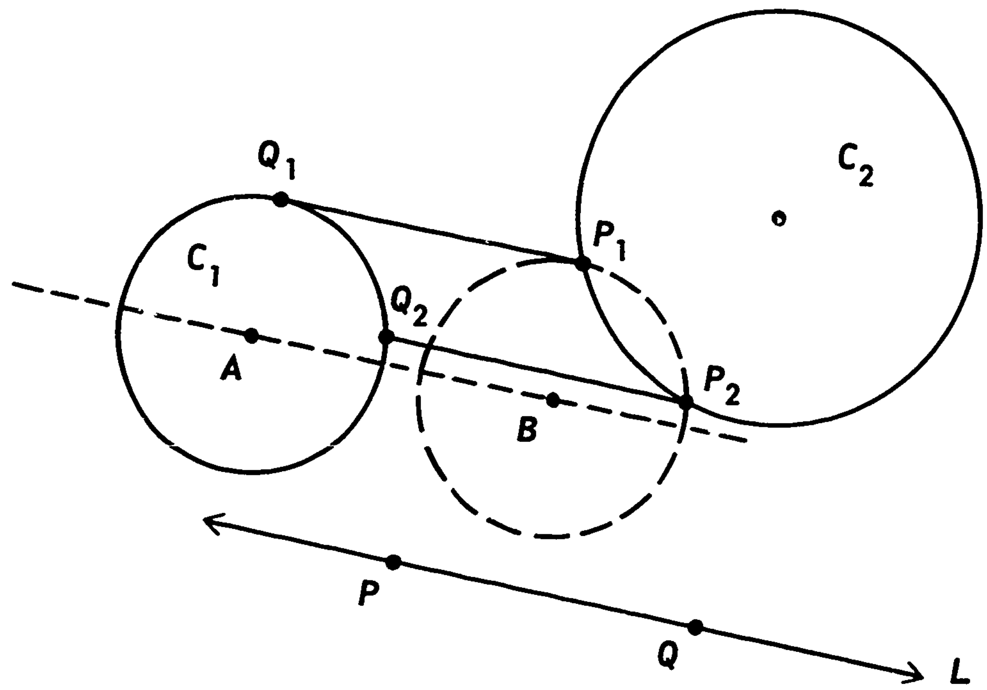


Fig. G-16.

6-5 SPECIAL KINDS OF RIGID MOTION

General discussion. In this section we give an informal description of various kinds of rigid motion that will be considered further in later sections. All rigid motions can be divided into two special and separate kinds: *direct* and *reversing*. The direct rigid motions include two special kinds: *translations* and *rotations*. We shall see in Section 6-10 (Theorem 6-1) that every direct rigid motion must either be a translation or a rotation. (The identity motion is usually counted as both a translation and a rotation.) The reversing rigid motions include two special kinds: *reflections* and *glide reflections*. In Section 6-10 (Theorem 6-2) we see that every reversing rigid motion must either be a reflection or a glide reflection. The present section gives the pupil a first introduction to the ideas: direct; reversing; translation; rotation; and reflection. Translations are considered in more detail in Section 6-7, rotations in Section 6-8, and reflections in Section 6-9. Section 6-10 gives further facts about these ideas, including the two theorems mentioned above. Glide reflections are not discussed in this section or in later sections except for a brief treatment in Section 6-10. The motion in Figure 47 is a glide reflection.

Although this section is short, and the ideas are easy, it is important that the student have a clear understanding of the new words given to him. You should allow two or possibly even three class meetings to discuss this material with your class and to go over the problems.

Warning. The new ideas introduced in this section are defined in terms of tracing paper movement. More purely mathematical definitions for translation, rotation, and reflection will be given in Sections 6-7, 6-8, and 6-9. The use of "clockwise order" and "counterclockwise order" helps to put the definitions of *direct* and *reversing* in somewhat more mathematical form, but the ideas "clockwise" and "counterclockwise" are still not perfectly mathematical. Perfectly mathematical definitions for *direct* and *reversing* can be given, but they require a long treatment and rather complex proofs to show that they are satisfactory. Since the underlying idea is very simple when thought of in terms of tracing paper, we shall, in Chapter 6, omit the purely mathematical definitions. If a pupil asks for a more mathematical definition, tell him that he can get definitions using "clockwise" and "counterclockwise" by taking the two statements before Problems 6-5 which are called *facts*, and thinking of them as *definitions*. If he does this, he is then left with the problem of proving, as a theorem, that every rigid motion must be either direct or reversing. The proof is not hard, but it is rather long and we do not give it here. It amounts to showing that if the vertices of some one triangle are carried from clockwise order to clockwise order by a rigid motion, then the vertices of any other triangle are carried from clockwise order to clockwise order by the same rigid motion.

Note. Direct motions are sometimes called "orientation-preserving," and reversing motions are sometimes called "orientation-reversing." Rigid motions in three dimensions can also be divided into direct and reversing, although the informal definitions are not quite as simple as for rigid motions in the plane. The book, *Through the Looking Glass* , by Lewis Carroll tells what it would be like to live in a world after it had been subjected to a reversing rigid motion.

Answers to
PROBLEMS 6-5

Student Text Pages 210-211

1. In 1 of Problems 6-2B, the motion is direct and is a translation.
   In 2 of Problems 6-2B, the motion is direct and is a rotation about the centre of the circle D.
   In 3 of Problems 6-2B, the motion is reversing and is a reflection about the line L_1.
   For 5 of Problems 6-2B, we use Figure 38 and the notation given in the Guide for the answer to Problem 6 of Problems 6-3.

$\binom{ABC}{DEF}$ is direct and a translation. $\binom{ABC}{EFD}$ is direct and a rotation about point P_1 in Figure G-11. $\binom{ABC}{FDE}$ is direct and a rotation about P_2 in Figure G-11. $\binom{ABC}{FED}$ is reversing and a reflection about line L in Figure G-11. $\binom{ABC}{EDF}$ is reversing but not a reflection. $\binom{ABC}{DFE}$ is reversing but not a reflection. (Both $\binom{ABC}{EDF}$ and $\binom{ABC}{DFE}$ are glide reflections.)

2. In Figure G-17 below, $Q_1 = T(Q)$ if T is direct, and $Q_2 = T(Q)$ if T is reversing. Note, as a general principle, that if we are given the images of two points, then there are exactly two motions possible, one of them direct and one of them reversing. See Problems 5 and 6 in Problems 6-2C.

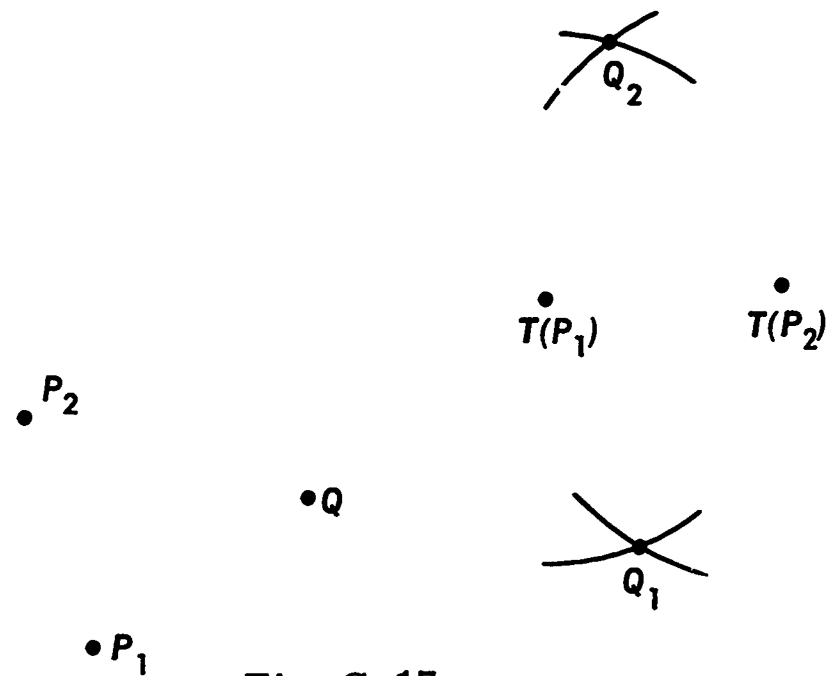


Fig. G-17.

In the figure above, the points Q_1 and Q_2 are found by drawing a circle about $T(P_1)$ with radius P_1Q, and a circle about $T(P_2)$ with radius P_2Q. Q_1 and Q_2 are the two intersections obtained. Q_1 is selected so that $T(P_1)$, $T(P_2)$, Q_1 occur in clockwise order (since P_1, P_2, Q occur in clockwise order.) Q_2 is selected so that $T(P_1)$, $T(P_2)$, Q_2 occur in counterclockwise order.

3. (a) Yes. (b) Yes. See the solution to Problem 2 above.

6-6 CONGRUENT FIGURES

General discussion. You should allow one or possibly two class meetings for the material in Section 6-6. There are no problems to be solved, and the text presents ideas that will already be fairly obvious to some of the pupils. The section is in two parts. The first part (up to the *Note on Isometric Mappings*) takes the ideas of Section 6-1 and restates them using the mathematical words of Section 6-3. The first part begins by noting that a triangle must be congruent to its image under any rigid motion (by SSS), and then showing that given any two congruent triangles, there is a rigid motion which makes one triangle the image of the other. Thus we see that two triangles are congruent if and only if there is a rigid motion which makes one triangle the image of the other. Finally, we use the idea of rigid motion to make a *general definition* of congruence for any pair of figures in the plane: we say that two figures are *congruent* if there is a rigid motion which makes one figure the image of the other. Because of our result above about triangles, we know that our now general definition of congruence agrees with our old definition for the special case of triangles.

In previous sections, we have taken "rigid motion" to mean "mapping given by a movement of tracing paper." In Section 6-3, we remarked that the idea of *isometric mapping* puts the idea of rigid motion in purely mathematical form. It was clear in Section 6-3 that every tracing paper movement gives an isometric mapping. Now, in the second part of Section 6-6, we complete the justification of our remarks in Section 6-3 by showing that for every isometric mapping there is a movement of tracing paper that gives that mapping. This is not a purely mathematical result, since it talks about tracing paper. Even though the result is not purely mathematical, a convincing argument (or "justification") for it can be given; and this is the argument given here in Section 6-6.

The argument refers to a construction in Section 6-2. This is the construction for Figures 29 and 30 in the Text. It begins with the words "Let *ABC* be a triangle." If you omitted this construction when you covered Section 6-2, and if you are not omitting the second part of Section 6-6, then you should go back and do this construction with your pupils now.

Note. The result about triangles in the first part of Section 6-6 uses the tracing paper idea of rigid motion. To get a purely mathematical proof of this result (using "rigid motion" to mean "isometric mapping"), proceed as follows. (1) Show that a triangle must be congruent to its image under any isometric mapping. (The proof is immediate by SSS exactly as before). (2) Given two congruent triangles, use the construction from Section 6-2 (mentioned above) to define a mapping which you can show to be isometric.

In comment (b), just before the *Note on Isometric Mappings*, the words "original study of congruent triangles" refer not to earlier portions of this chapter, but rather to work in a previous year where the pupil was first introduced to the idea of congruent triangles.

6-7 TRANSLATIONS.

General discussion. Section 6-7 on translations, Section 6-8 on rotations, and Section 6-9 on reflections each follow the same general outline. First, a particular kind of construction is presented. (In Section 6-7, this is the construction for "translating" a point P by a given vector **U**. In Section 6-8, this is the construction for "rotating" a point P through a given angle β about a given point O. In Section 6-9 this is the construction for "reflecting" a point P in a given line L.) A construction of this kind (with a fixed **U** in the case of 6-7, a fixed β and O in the case of 6-8, a fixed L in the case of 6-9) gives a mapping from points of the plane to points of the plane. Second, a proof is given that this mapping must be isometric and hence a rigid motion. (As we mentioned in the Introduction, the proof in this second part can be omitted or treated lightly.) Third, this kind of mapping is given a name (*translation* in 6-7, *rotation* in 6-8, *reflection* in 6-9). Fourth, various facts about mappings of this kind are given. These include some geometrical facts (like, for example, the fact that every line gets moved onto itself or else parallel to itself by a translation), some facts about coordinate axes, and, in the case of rotations and reflections some facts about symmetry.

In case pupils have already done some coordinate geometry, you will find that the parts on coordinate axes can be treated rather briefly. Otherwise (unless you omit the parts on coordinate axes) the meaning of locating a point P on the plane by $P(a,b)$ should be discussed in more detail. Once the meaning of (a,b) has been made clear, the solutions to the problems on coordinate axes will become quite simple.

Note. In 6-7, the Text asks the question: "For what lines is it true that the image of L is L itself?" The answer is: "those lines which are parallel to **U**."

You should allow about three class meetings to cover the material and problems in Section 6-7.

1. In the following figure, A is the image of the triangle, B is the image of the circle, and C is the figure whose image is the circle.

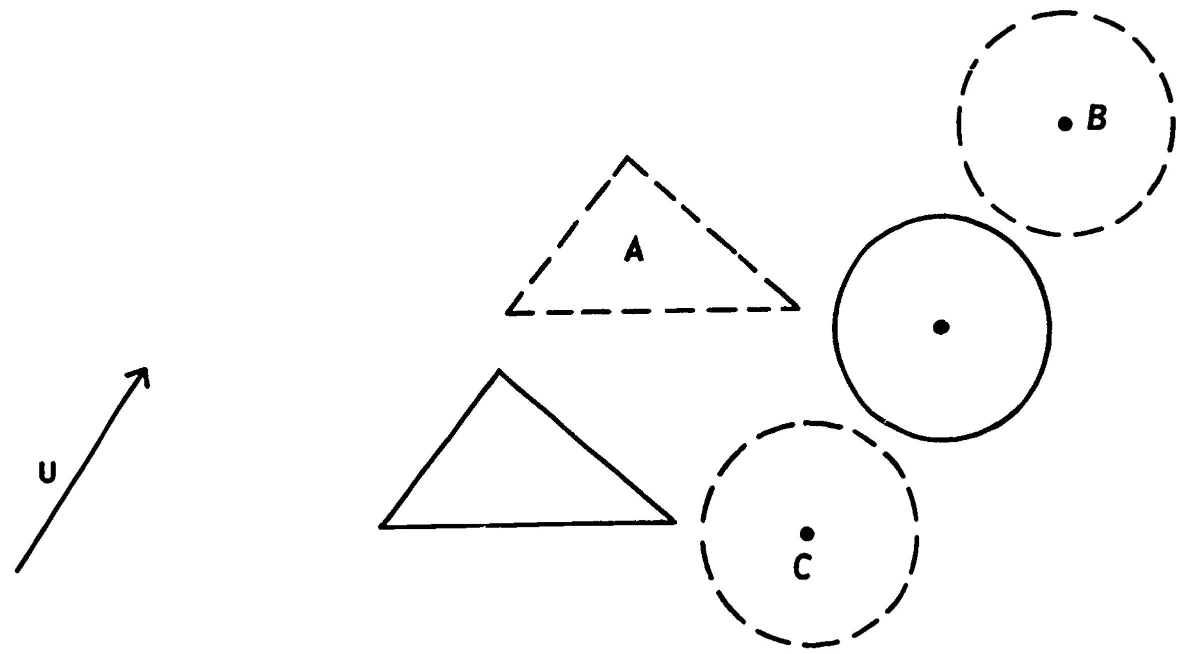


Fig. G-18.

2. All lines parallel to **U**.

3. The identity motion.

4. The result is a translation by the vector **W** given in the following Figure G-19.

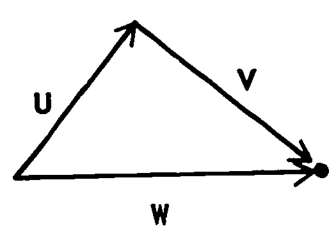


Fig. G-19.

5. (3, -3), (0, 0), (9, -14).

6. In the following figure, right triangle I is congruent to right triangle II

(by ASA), hence the two legs of II have lengths m (for the horizontal leg), and n (for the vertical leg), and hence P' has the coordinates $(a + m, b + n)$.

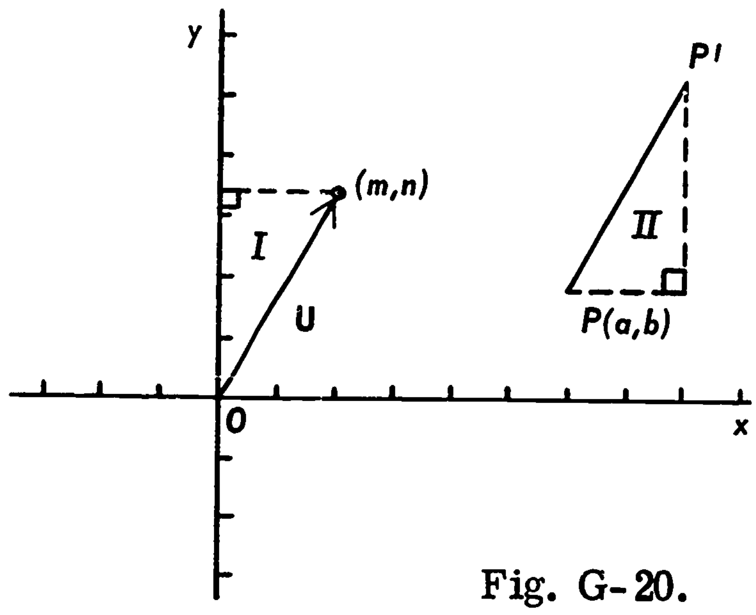


Fig. G-20.

7. Let P be any point and let P' be $T(P)$. Let **U** be the vector from P to P'. Let Q be any point, and let $Q' = T(Q)$. By assumption $PP' = QQ'$. Since T is a rigid motion, we also have $PQ = P'Q'$. It follows that one of the two possibilities given in Figure G-21 must occur.

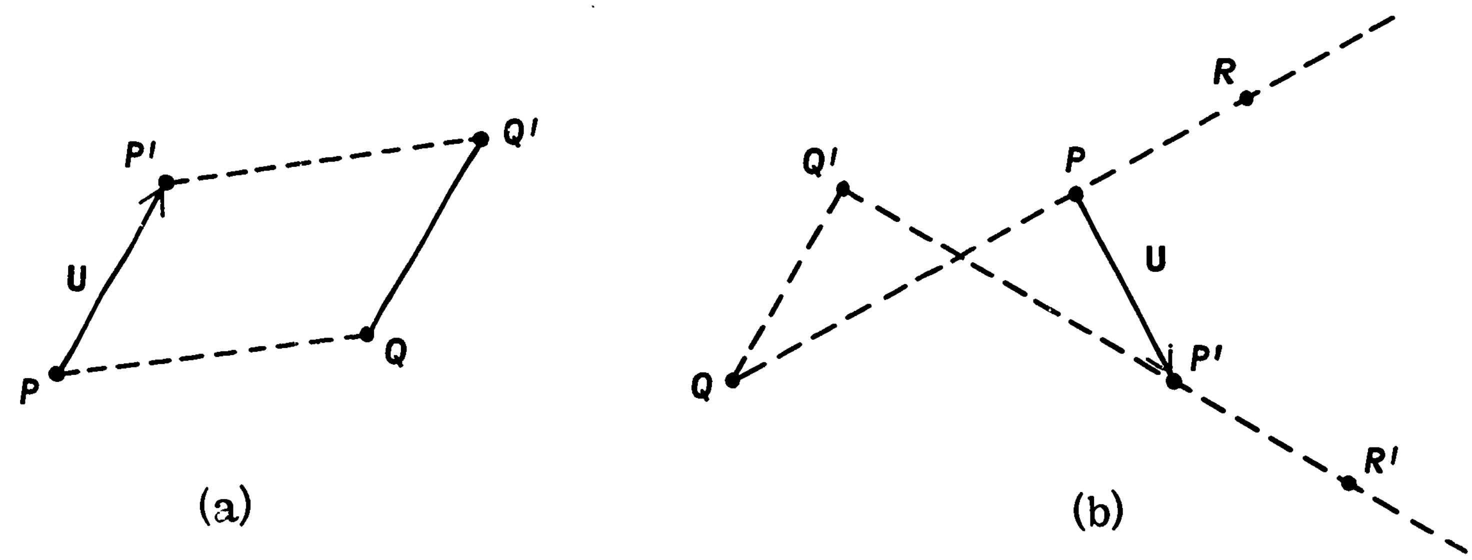


Fig. G-21.

We next show that possibility (b) cannot occur. Let R be any point far out along line $\overleftrightarrow{PQ}$ on the side of P opposite to Q. Then $R' = T(R)$ must be a point far out along $\overleftrightarrow{P'Q'}$ on the side of P' opposite to Q'. As we take R farther and farther out, the distance RR' continues to increase. This is contrary to our assumption that all points move the same distance. Thus the only possibility is (a). Since a simple quadrilateral with opposite sides equal in pairs must be a parallelogram, we have $\overline{QQ'}$

parallel to $\overline{PP'}$. But this means that Q' is the result of translating Q by the vector **U**. Since Q was any point, we see that T is a translation. (Note that Problem 7, together with the facts given in the text, tells us that the translations are exactly those rigid motions in which every point moves the same distance. We therefore say that the translations can be *characterized* as those rigid motions which move every point the same distance.)

8. Consider a rotation of the tracing paper through 180° about some fixed point Q. This motion is not a translation since not every point is moved the same distance. (In fact one point, namely Q, is not moved at all.)

9. Let the points be given as in the following figure. Let T be the result of a reflection in L followed by a translation by **U**. T will then carry P_1 to P_1' and P_2 to P_2'. The motion T is reversing (since the tracing paper must be turned over to carry it out). Hence T cannot be a translation.

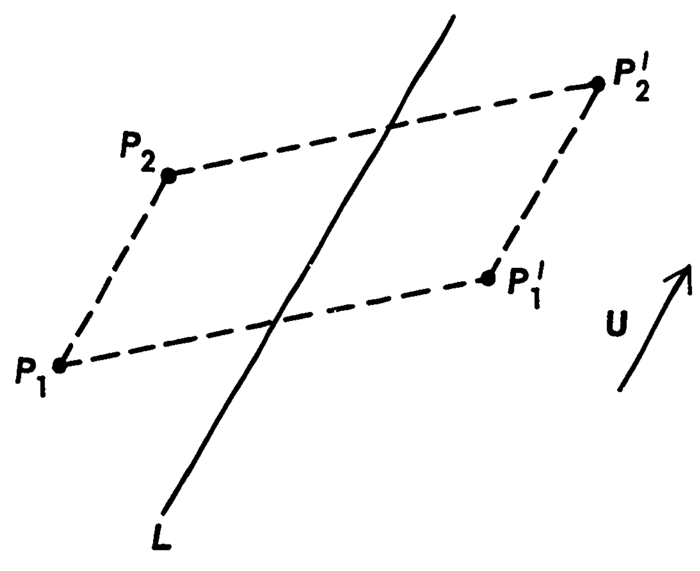


Fig. G-22.

10. Let Q be any third point. By Problem 5 of Problems 6-2C, T must carry Q to either Q' or Q'' in the following figure. Because T is direct, it must carry Q to Q'. But triangle P_1P_2Q is congruent to triangle $P_1'P_2'Q'$ (by SSS). Hence, using equal angles from these triangles, we get $\overline{P_1Q}$ parallel to $\overline{P_1'Q'}$. Since $P_1Q = P_1'Q'$, we have that $P_1QQ'P_1'$ is a parallelogram (because a simple quadrilateral with one pair of opposite sides equal and parallel must be a parallelogram). Hence $QQ' = P_1P_1'$. But Q was any point. Hence T moves every point

the same distance. Hence, by Problem 7, T must be a translation.

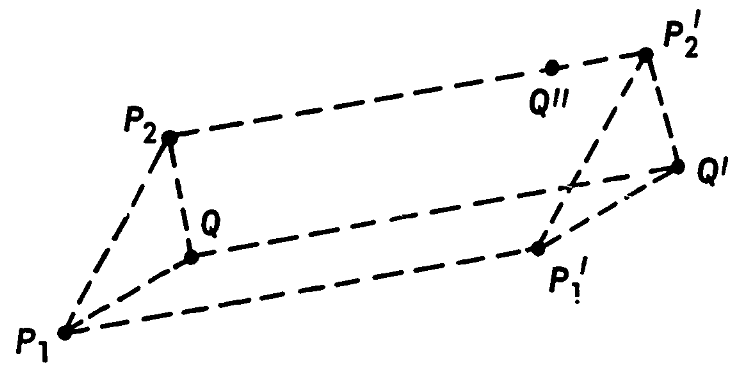


Fig. G-23.

Note. In the answers to Problems 7 and 10, "simple quadrilateral" means quadrilateral in which no pair of opposite sides intersects.

6-8 ROTATIONS.

General discussion. The text for Section 6-8 follows the same general outline as for Section 6-7. See the general discussion above for Section 6-7. Note that we pay special attention in the problems to *quarter-turns* and *half-turns*. Note also that Problem 12 of Problems 6-8A asks for a proof for the "fundamental fact" mentioned just before Problems 6-8A. (See below for a solution.)

After Problems 6-8A, rotational symmetry is discussed.

The idea of symmetry is quite important in bringing out geometrical properties of figures. For example in dealing with the equilateral triangle, we could infer that the centre is the same distance from each vertex, in case the pupils had not come across this fact previously, and so on.

We see that the square cannot be brought into coincidence with itself by a rotation of 120°, however a rotation of 90° does.

In Figure 74, rotations of 60°, 120°, 180°, 240° will bring the diagram into coincidence with itself. The idea of *order of symmetry* can be introduced at this stage. However the text does not choose to emphasize this idea. Figure 75 does not have rotational symmetry about O.

In connection with coordinate axes, note that, since a half-turn would take a point distant r along the positive x-axis to a point distant r along the negative x-axis and similarly for a point on the y-axis, *a half-turn would send* *a point* $P(a,b)$ *to point* $P'(-a, -b)$. A quarter turn would send a point $P(a,b)$ to point $P'(-b,a)$.

You should allow three or four class meetings to cover the material and problems in Section 6-8.

Answers to PROBLEMS 6-8A

Student Text Pages 222-225

1. $m(A\hat{O}A') = 90°$
   $\overline{OA} \equiv \overline{OA'}$

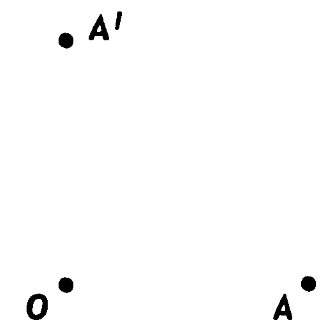


Fig. G-24.

2. BXB' is a straight line
   $\overline{BX} \equiv \overline{B'X}$

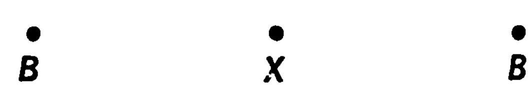


Fig. G-25.

3. $B'A'$ is at right angles to $\overline{AB}$
   $\overline{AB} \equiv \overline{A'B'}$

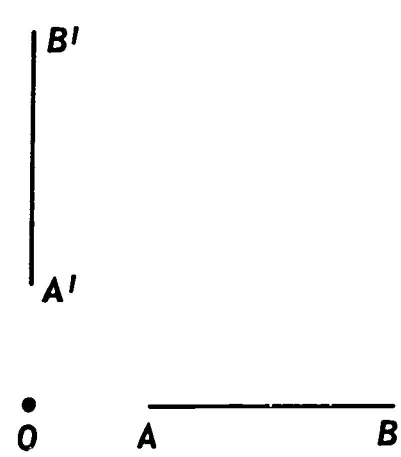


Fig. G-26.

4. B', A', O, A, B, all lie on a straight line.
   $\overline{AB} \equiv \overline{A'B'}$
   Move from right to left to get from A' to B'.

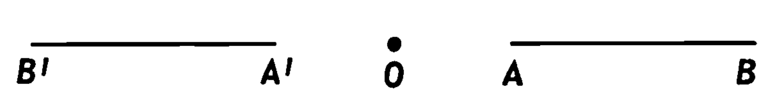


Fig. G-27.

5.

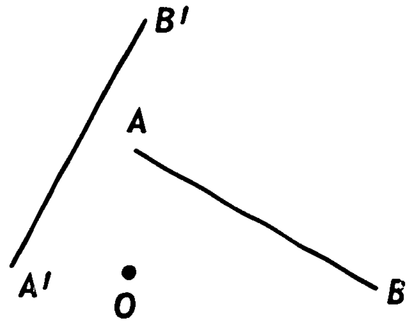


Fig. G-28.

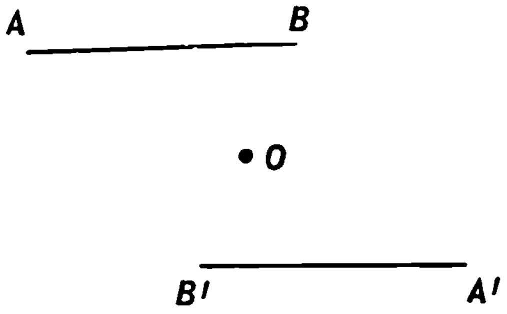
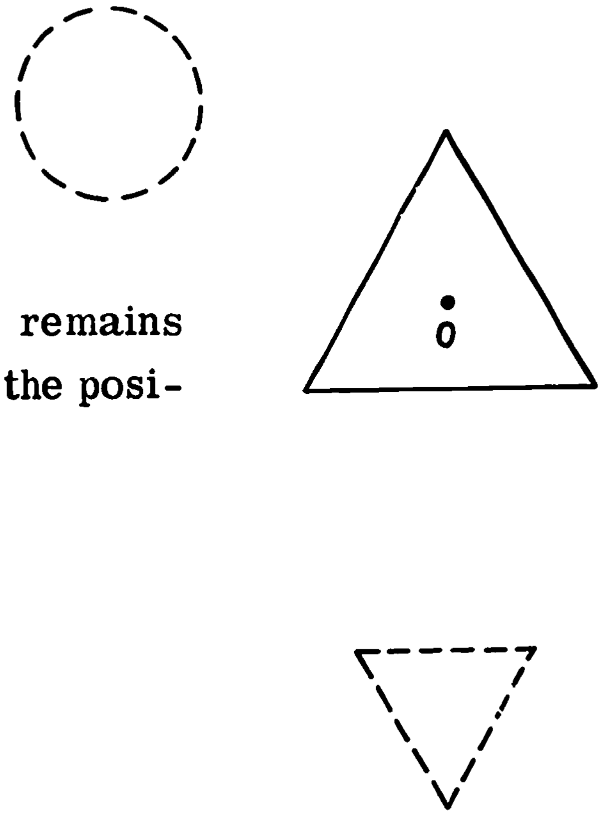


Fig. G-29.

6. Join $\overline{AB'}$, $\overline{BA'}$ to get the quadrilateral $AB'A'B$.

BOB', AOA' are straight lines.
We prove $\Delta AOB \equiv \Delta A'OB'$.
[Since $\overline{OA} \equiv \overline{OA'}$; $\overline{OB} \equiv \overline{OB'}$; $\overline{AB} \equiv \overline{A'B'}$.]
Hence $A\widehat{B}B'$ (same as $A\widehat{B}O$) $\equiv A'\widehat{B'}B$ (same as $A'\widehat{B'}O$).
So $\overline{AB}$ is congruent and parallel to $\overline{B'A'}$.
Therefore $AB'A'B$ is a parallelogram.



7. The equilateral triangle with centre O remains invariant. The other figures move to the positions indicated by the broken lines.

Fig. G-30.

8. O lies at the mid point of $\overline{AA'}$.
9. O lies on the perpendicular bisector of $\overline{AA'}$.

Challenge Problem

10. If rotation is about O, $\overline{OA} \equiv \overline{OA'}$ and $\overline{OB} \equiv \overline{OB'}$.
For $\overline{OA} \equiv \overline{OA'}$, the locus of O is the perpendicular bisector of $\overline{AA'}$.
For $\overline{OB} \equiv \overline{OB'}$, the locus of O is the perpendicular bisector of $\overline{BB'}$.
So O will be given by the intersection of the perpendicular bisectors of $\overline{AA'}$ and $\overline{BB'}$. There is a rigid motion which carries $\overline{AB}$ to $\overline{A'B'}$ and is not a rotation, namely the reversing motion which takes A to A' and B to B'.

11. By similar reasoning as in Problem 10, we get O to be given by the point of intersection of the perpendicular bisectors of $\overline{AA'}$, $\overline{BB'}$ (and $\overline{CC'}$). All three are concurrent. (Note that, unlike Problem 10, there is no other motion which takes ΔABC to $\Delta A'B'C'$. See Problems 5 and 6 of Problems 6-2C.)

12. Let L be a given line and let L' be its image under a rotation about O. Make the construction indicated in the following figure by dropping perpendiculars from O to L and L'. Then $m(C\hat{O}B) + m(O\hat{B}C) = 90°$ and $m(D\hat{A}B) + m(O\hat{B}C) = 90°$. Hence $m(C\hat{O}B) = m(D\hat{A}B)$, which is the fact to be proved.

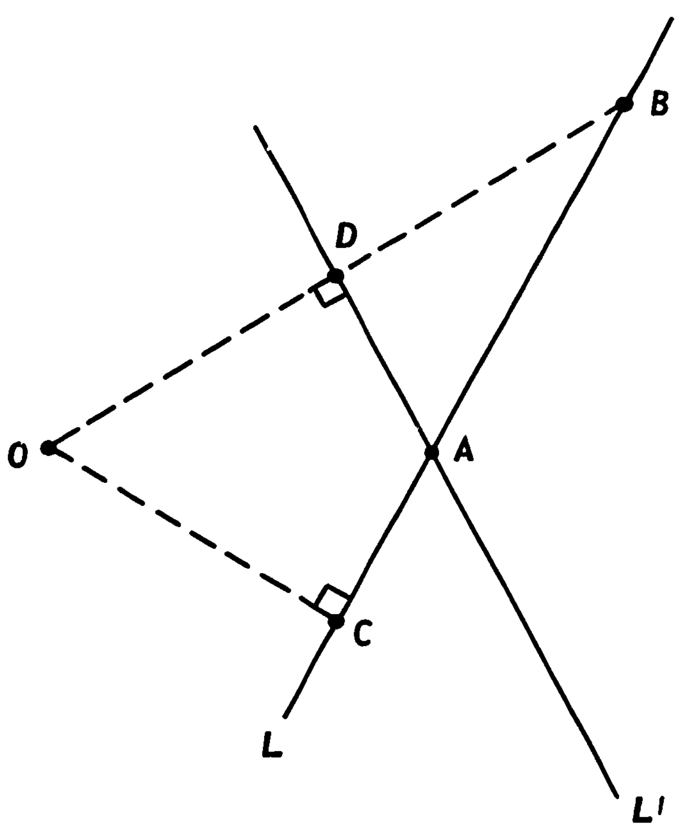


Fig. G-31.

1. All the figures are regular except the triangle, so each has rotational symmetry except the triangle.

2. The regular octagon (1st figure) has order of symmetry 8.
The regular pentagon (2nd figure) has order of symmetry 5.
The regular hexagon (3rd figure) has order of symmetry 6.
The triangle has no rotational symmetry and so its order of symmetry is 1. (The identity motion is the only rotation carrying this triangle onto itself.)
The square has order of symmetry 4.

3. Let O be a fixed point of the given rigid motion. If every other point is also a fixed point, then we have the identity motion which is a rotation. Otherwise let P be a point which is not a fixed point. Let $P' = T(P)$. Let $\beta = m(P\hat{O}P')$. Let Q be any other point. (See Figure G-32 below). By the isometric property, $T(Q)$ must be either Q' or Q''. Since T is direct, $T(Q)$ must be Q'. By SSS, $\Delta\, OPQ$ is congruent to $\Delta\, OP'Q'$. Therefore (by a simple equation) we get $m(Q\hat{O}Q') = m(P\hat{O}P') = \beta$. Thus Q' is obtained from Q by a rotation through the angle β. Since Q was any point, we have that T is a rotation.

Note. The solution to Problem 3 puts us very close to a proof of Theorem 6-1 (in Section 6-10) which states that every direct rigid motion is either a rotation of a translation. To prove Theorem 6-1, it only remains to show that any direct motion without fixed points must be a translation. For more on this, see the discussion below on Section 6-10.

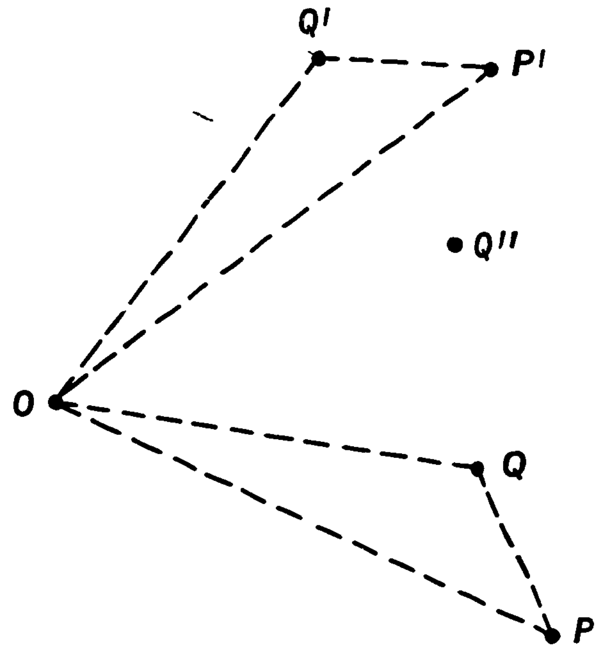


Fig. G-32.

6-9 REFLECTIONS.

General discussion. The text for Section 6-9 follows the same general outline as for Section 6-7. See the general discussion above for Section 6-7.

Although some of the pupils will have come across reflections in Physics, it would not be a waste of time to go through it thoroughly again. The idea of reflection along a line instead of a plane mirror may not be easy to get across initially. The construction given in the definition of reflection is the most practical for obtaining images of reflection. The pupils' attention must be drawn to the fact that this construction is equivalent to turning the tracing paper over along the line of reflection.

After problems 6-9A, the idea of *axis of symmetry* is discussed. Another way of stating Definition 6-4 is as follows. "If a figure is invariant under a reflection, then the line of reflection is called an *axis of symmetry* of the figure."

You should allow about three class meetings to cover the material and problems in Section 6-9.

Answers to
PROBLEMS 6-9A

Student Text Pages 230-232

1. Yes
2. Same direction as $\overrightarrow{AB}$, i.e., left to right.

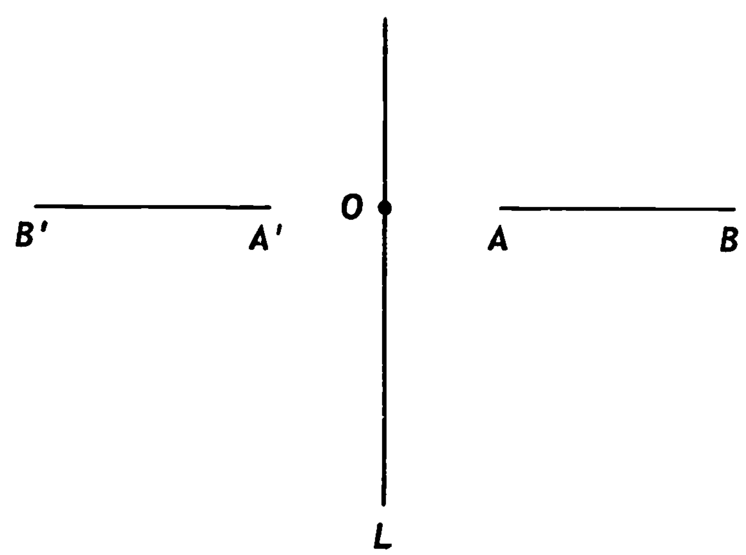


Fig. G-33.

4. Right to left.

5. Yes.

6. $\Delta ABC \equiv \Delta A'B'C'$
Because reflection is an isometric transformation.

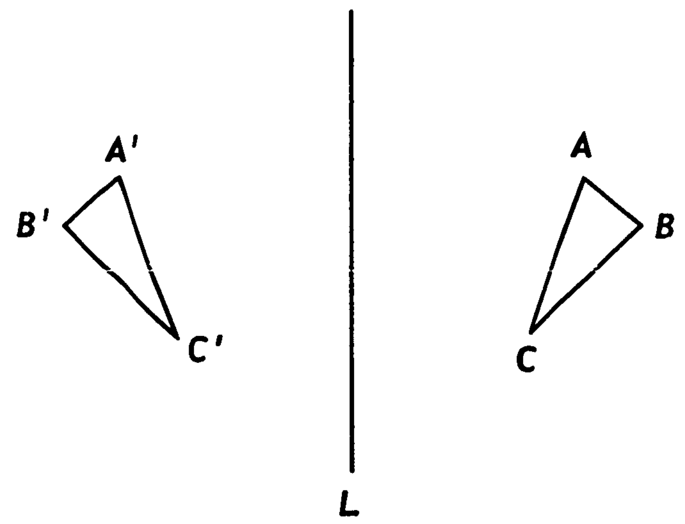


Fig. G-34.

7. Yes.

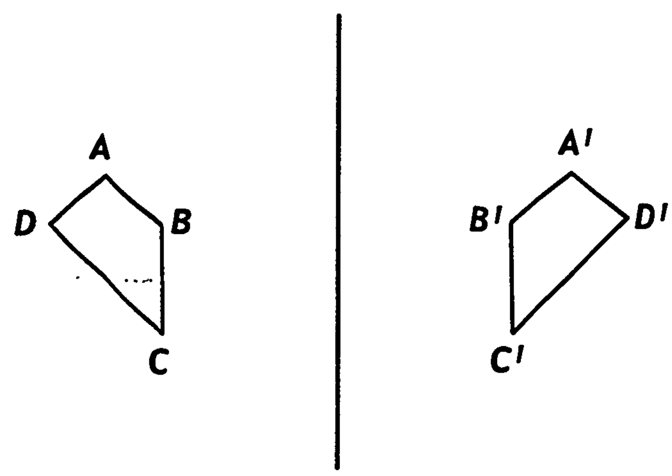


Fig. G-35.

8. Image is a circle. Arrow in image will point counterclockwise.

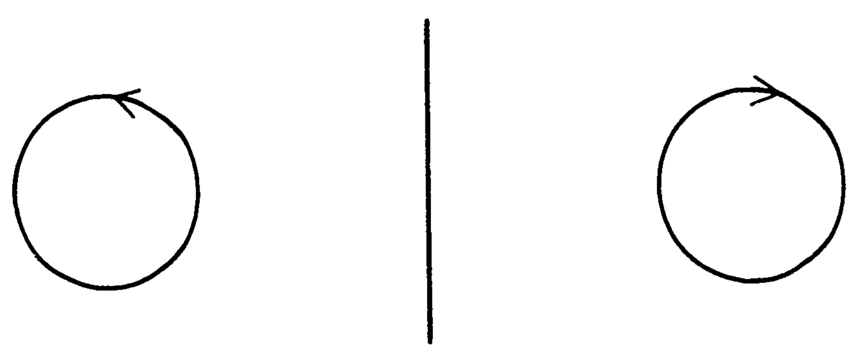


Fig. G-36.

9. The line of reflection is obtained by joining the mid-points of $\overline{AA'}$ and $\overline{BB'}$ in each case.

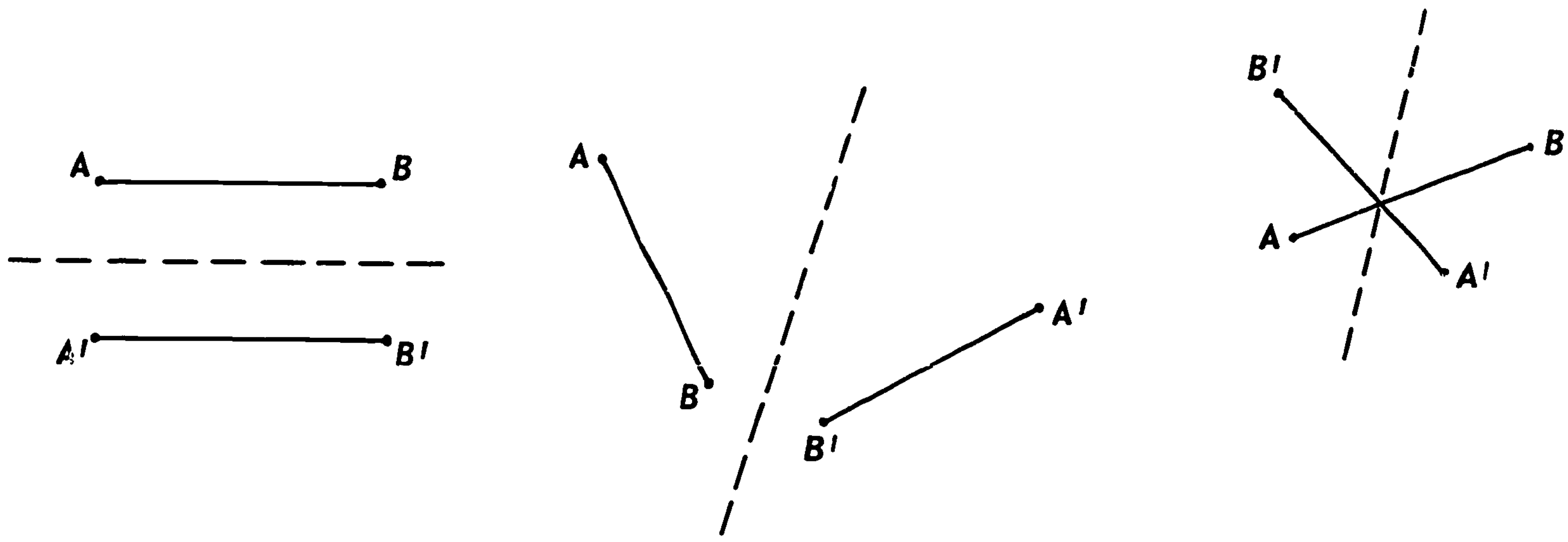


Fig. G-37.

10.

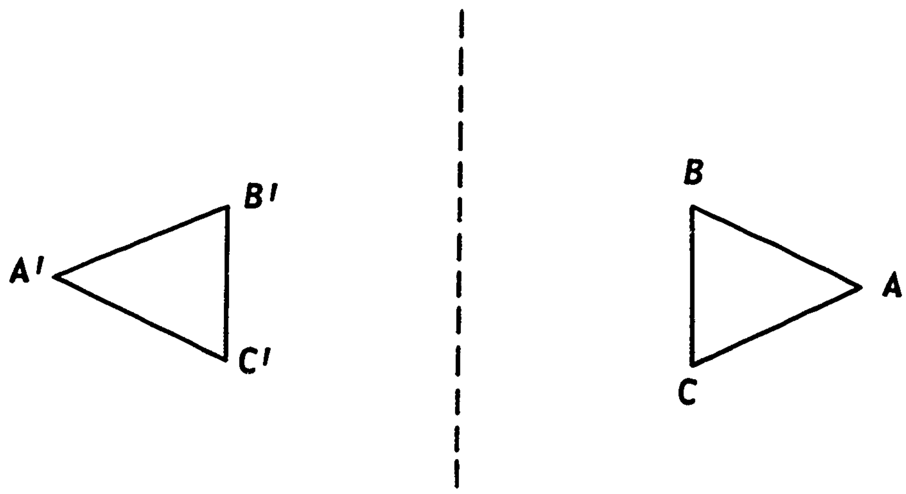


Fig. G-38.

11. Not possible.

It is not a reflection since the image shown in the problem results from a direct rigid motion while a reflection is reversing.

Answers to
PROBLEMS 6-9B

Student Text Page 234

1. Through every point X of $\overleftrightarrow{AB}$, we drop a perpendicular to L, which cuts L at P and produce the perpendicular to cut $\overleftrightarrow{CD}$ at X'. $\overline{XP} \equiv \overline{X'P}$. (This is easily shown by congruent triangles.) Alternately, from any point Y on $\overleftrightarrow{CD}$ we can get Y' on $\overleftrightarrow{AB}$ in the same way as above with $\overline{YP} \equiv \overline{Y'P}$. So we have shown that every point of $\overleftrightarrow{AB}$ has its image on $\overleftrightarrow{CD}$ and vice

versa. Hence L is an axis of symmetry.

Line m is also an axis of symmetry.

This is enough to show that L and m are the loci of points equidistant from $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$.

2. A half-turn of P about O gives P' where $P'OP$ is a straight line and $\overline{OP} \equiv \overline{OP'}$. This is the same as the result given in the problem.

Challenge Problem

3. $y = x$ is an axis of symmetry of this diagram. So P' is the point (b,a).

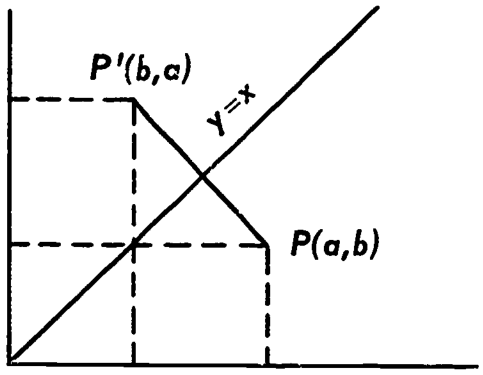


Fig. G-39.

6-10 ONE MOTION FOLLOWED BY ANOTHER.

General discussion. You should allow five or six class meetings to cover the material in this section. The first two class meetings would cover up through Problems 6-10B. The third and fourth class meetings would cover up through Problems 6-10D. The fifth and possibly sixth class meetings would cover the material given under the heading "A Basic Theorem."

The material given in Section 6-10 provides only the briefest and shallowest introduction to a rich mathematical subject. A full year could easily be spent in further and deeper study of some of the ideas that are touched on here. The basic idea is that one rigid motion can be *combined* with another by carrying out the two motions one after the other. This combination yields a resulting mapping which is, by itself, a rigid motion. We say that this third rigid motion is the "result of combining" the first two rigid motions. If the first two rigid motions are R and S and if T is the third rigid motion which results from carrying out first R and then S, we say that "T equals R *followed by* S." The act of combining two rigid motions is sometimes called an *operation*. It is a way of putting two motions together to get a motion, just as, in arithmetic, the *operations* of addition and multiplication are ways of putting two numbers together to get a number.

In Section 6-10, we first define the operation of combining rigid motions, and then we look at some facts concerning some of the simplest combinations of different kinds of rigid motion. This material is covered in Problems 6-10A on translations combined with translations, Problems 6-10B on translations combined with half-turns, Problems 6-10C on half-turns combined with half-turns, and Problems 6-10D on reflections combined with reflections in the special case where the two lines of reflection are parallel. Finally, in the last part of the section, we give some general facts and rules about combinations of rigid motions. (Such as, for example, the rule that any combination of two reversing motions must be a direct motion.)

If we were to carry out further and deeper study of the operation of combining rigid motions, there are two directions in which we could proceed. (1) We could look at ways in which this operation (and facts about it) can be used to solve geometrical problems. Section 6-11 gives some examples of this (Examples 3 and 4) where we see that the use of this operation leads to quicker and easier solutions than we would otherwise get. (2) We could study the "algebra" of this operation in much the same way as, in previous years, the pupil has studied the algebra of the addition and multiplication operations from arithmetic. Except for a few brief comments in some of the problems, Chapter 6 does not take up this "algebra" of rigid motions. In *Secondary Five*, both in algebra and geometry, the pupil will study it further. The rigid motions form what is called, in higher algebra, a *group*.

Comments. In considering Definition 6-6, remind the student that a rigid motion is a *mapping* given by a tracing paper movement. It is not the actual path of the movement. This is important if the student is to understand, for example, how the combination of two reflections (in Figure 101) can be a translation.

Note that a number of facts are given without proof. (For example, the fact that a translation followed by a translation must be a translation.) In most of these cases, proofs are easy to find, and can be given to your pupils as added exercises. (For example, use of Problem 7 in Problems 6-7, together with the fact that each translation moves all points the same distance, gives an immediate proof that a translation followed by a translation must be a translation.)

Algebraic notation for the operation of combining rigid motions is introduced, to a small extent, in some of the problems. (Problem 2 of Problems 6-10C is the first of these.)

Warning. Note that, in the algebraic notation used in Chapter 6, the motion R followed by S is written as SR. Thus we abbreviate "R followed by S" by having R follow S on the page. This reversal is in some ways bad and can cause confusion to the pupil. It has, however, the following advantage, which explains why we use it: if T is R followed by S, then for any point P, $T(P) = S(R(P))$ If we write $T = SR$, then we can abbreviate $S(R(P))$ as $SR(P)$. For us, the convenience of this notation outweighs the dangers of confusion.

Note that RS need not be the same as SR. (In the special case where R and S are both translations, it is true that $RS = SR$.) Problems 6-10B, 6-10C, and 6-10D give various examples where RS and SR are not the same rigid motion. Thus, to use a word from algebra, we would say that the operation of combining rigid motions is not *commutative*. (It is, however, *associative*, that is to say, $(QR)S = Q(RS)$; because, for any point P, both $[(QR)S]\ (P)$ and $[Q(RS)]\ (P)$ are the same as $Q\Big(R\big(S(P)\big)\Big)$.

The basic facts and rules given at the end of the section (under the heading "A Basic Theorem"), are not proved, although the first of these, the rule on direct and reversing motions, is immediately clear from the tracing paper idea of motion. These facts and rules are, however, an important and central part of Chapter 6. Please make every effort to persuade your pupils that they are important and true.

How could a proof for Theorem 6-1 be found? It would amount to showing that the method illustrated in Figure 105 always worked (that is to say, that if two of the perpendicular bisectors meet, then all three meet at a common point—the fact that we then have a rotation is immediate). This proof is based on congruent triangles. It is lengthy, but not hard.

What about Theorem 6-2? Here the construction (corresponding to the construction in Figure 105 for Theorem 6-1) is as follows. Take the midpoints of $\overline{AA'}$, $\overline{BB'}$, and $\overline{CC'}$. At least two of these midpoints are distinct (otherwise we have a half-turn, which is not reversing). All three midpoints are on the same line (a lengthy congruent triangles proof). Then T is either a reflection, or else a glide reflection, with this line as its line of reflection.

Answers to questions raised in Text. In the text on "Rotation Followed by Rotation," how do we show that the combination of two half-turns about the same point is the identity? Since a point gets rotated 180° by the first turn and then 180° by the second turn, it must be returned to its original position. The combination leaves every point fixed and we have the identity.

In the text for Figure 105, why must at least two of the perpendicular bisector lines be distinct? If all three coincide, it is easy to see (draw a figure) that the given motion has to be reversing, contrary to assumption.

Answers to
PROBLEMS 6-10A

Student Text Pages 238-239

To describe a translation completely we had to introduce vectors. Combining translations is, in effect, addition of vectors. We see in the diagram below that $\mathbf{U} + \mathbf{V} = \mathbf{W} = \mathbf{V} + \mathbf{U}$. That is, translation **U** followed by translation **V** gives translation **W**, which is also obtained from translation **V** followed by translation **U**.

It is a good idea to get the pupils to try several pairs of translations to demonstrate the rule for combining translations.

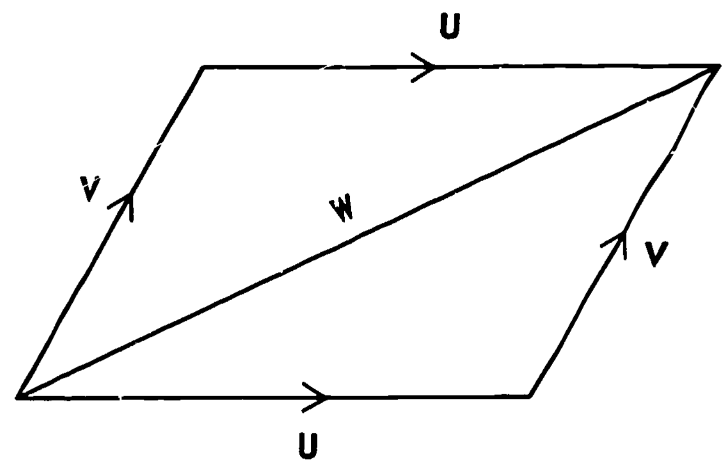


Fig. G-40.

1. The two sides of any triangle whose third side is **W** give a solution. (Of course, **U** and **V** are not included.) Hence there are an unlimited number of acceptable answers.

2. **W** is the single vector and **W** = **U** + **V**.

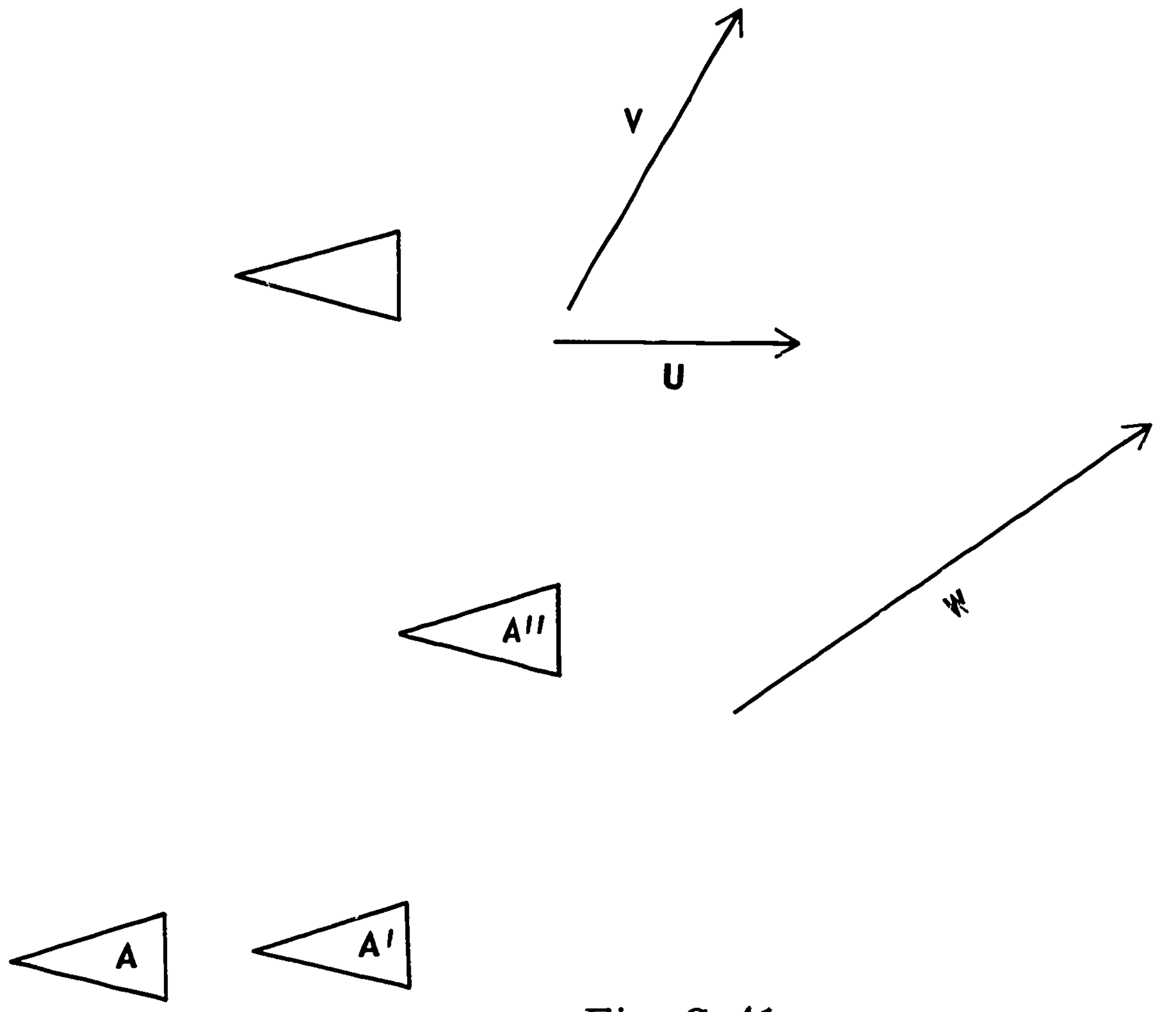


Fig. G-41.

3.

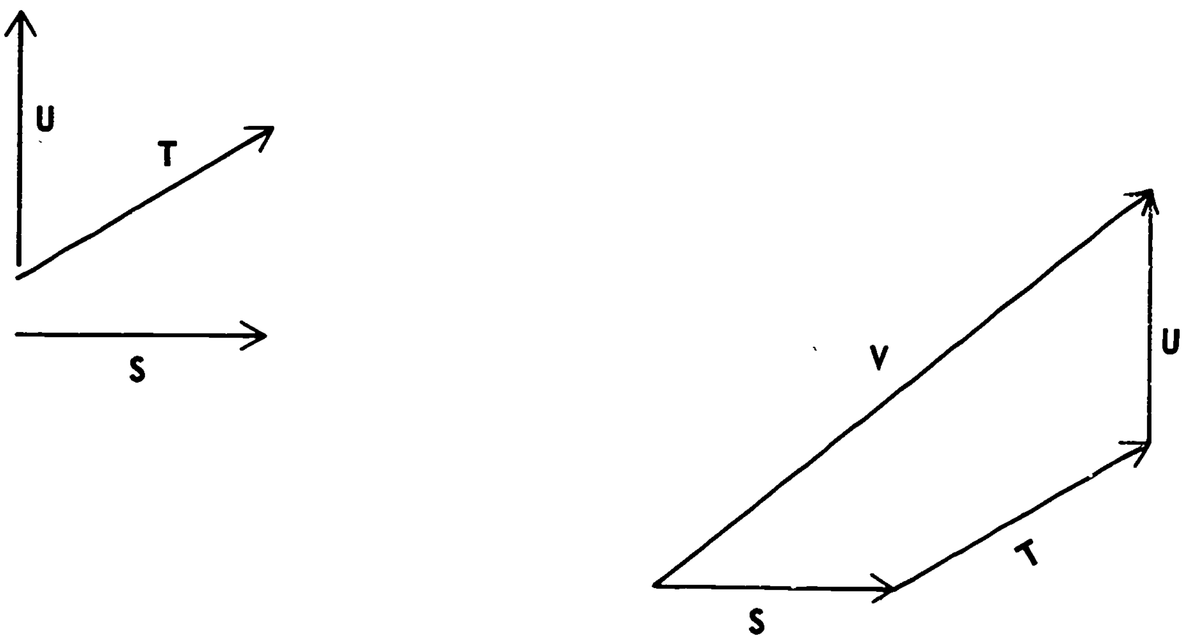


Fig. G-42.

4. Any right-angled triangle having **V** as hypotenuse will be an acceptable solution.

5. T will move A' to A'', 2T away from A.
T will move A'' to A'''. 3T away from A.

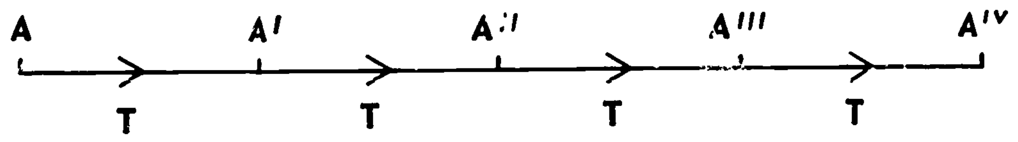


Fig. G-43.

In the preceeding problems try to see if the students can give a general solution.

Answers to
PROBLEMS 6-10B

Student Text Page 239

1. No. (Note that $C''=D'$ and that $D''=C'$. Note further that only a translation or a reflection will take C'' back to C and D'' back to D.

2. Yes.
For example: Take a point P along $\overline{HK}$ distant d from $\overline{CD}$. A half-turn about P followed by a translation **T** parallel to **W**, where the length of **T** is $W - 2d$, will move $\overline{CD}$ to $\overline{C'D'}$.

3. The final position will be at a distance W from $\overline{CD}$ but on the opposite side of $\overline{CD}$ from $\overline{C'D'}$. Not the same as $\overline{CD}$.

1.

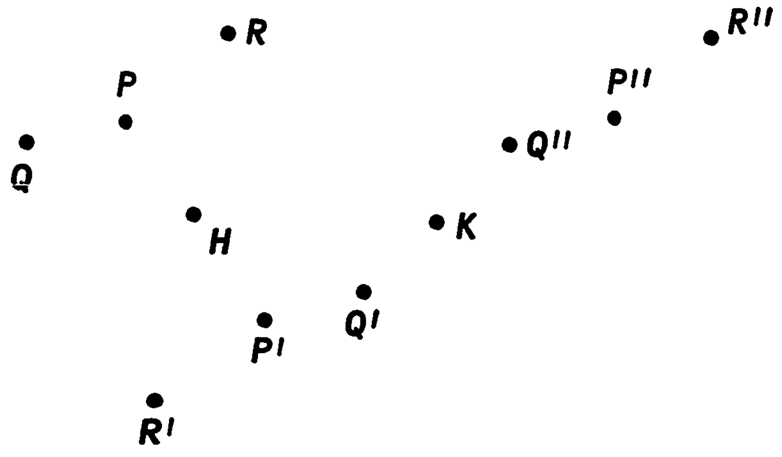


Fig. G-44.

$\overline{PP''} \equiv \overline{QQ''} \equiv \overline{RR''} = (2\overline{HK})$

They are parallel to each other, since each is parallel to $\overline{HK}$.

2. From Problem 1, GF is a translation given by $2\overrightarrow{GF}$, i.e., a translation parallel to $\overline{GF}$ and twice the length of $\overline{GF}$.
So $S = GF$,
since $\overrightarrow{LL'} = 2\overrightarrow{GF}$.

3. Calling the half-turn about A, A, half-turn about B, B, etc.
A followed by $B = \mathbf{T}$ where $\mathbf{T} = 2\overrightarrow{AB}$.
D followed by $C = \mathbf{S}$ where $\mathbf{S} = 2\overrightarrow{DC}$.
But $\overrightarrow{AB} = \overrightarrow{DC}$ ($ABCD$ is a parallelogram.)
Hence $\mathbf{T} = \mathbf{S}$,
i.e., $AB = DC$.

4. HK is a half-turn about K followed by a half-turn about H.
$HK \neq KH$.
The translation HK is equal in length to the translation KH but opposite in direction.

5. Define KJH as a half-turn about H, followed by a half-turn about J followed by a half-turn about K.

Suppose H carries P to P'
J carries P' to P''
K carries P'' to P'''

So KJH carries P to P'''.

KJ carries P' to P''' so $\overrightarrow{P'P'''} = 2\overrightarrow{JK}$.

JH carries P to P'' so $\overrightarrow{PP''} = 2\overrightarrow{HJ}$.

$\overline{P'P'''}$ is equal and parallel to $\overline{PP''}$

(because $\overline{JK} \equiv \overline{HJ}$).

Hence $PP''P'''P'$ is a parallelogram.

So $\overline{PJ} \equiv \overline{JP'''}$ (intersection of diagonals).

Hence J carries P to P''',

i.e., $KJH = J$.

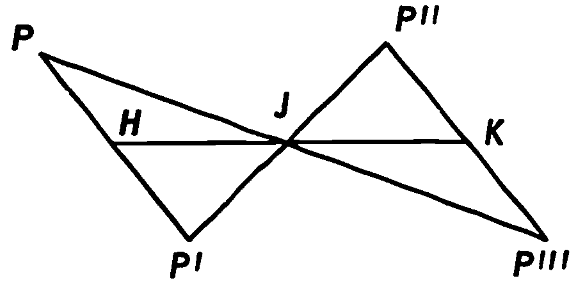


Fig. G-45.

6. $P(B) = B'$.

$Q(B') = B''$.

$S(B'') = B'''$.

$\overrightarrow{BB''} = 2\overrightarrow{PQ}$.

$\overline{PQ} \equiv \overline{RS}$ and $PQRS$ is a straight line.

Hence $BB'' = 2RS$ and $\overline{BB''}$ is parallel to $\overline{RS}$.

Since $\overline{B''S} \equiv \overline{SB'''}$, we have in $\Delta\, \overline{BB'B'''}$ that point R lies on $\overline{BB'''}$ and that $\overline{BR} \equiv \overline{RB'''}$.

Hence $R(B) = B'''$,

i.e. $SQP(B) = R(B)$.

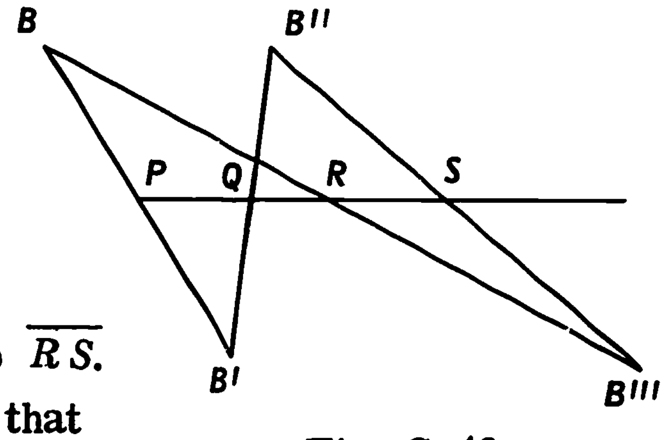


Fig. G-46.

7. $A(P) = P'$.

$BA(P) = P''$.

$C(P'') = P'''$.

In $\Delta\, PP'P''$,

$\overrightarrow{PP''} = 2\overrightarrow{AB}$ ($= 2\overrightarrow{DC}$),

since $ABCD$ is a parallelogram.

In $\Delta\, PP''P'''$, PDP''' is a straight line,

since $PP'' = 2\,DC$ and parallel to it.

So $PD = DP'''$.

Hence $D(P) = P'''$,

i.e.,$CBA(P) = D(P)$.

Also, $CD(P) = P_1$,

where $PP_1 = 2\,DC$ and parallel to it.

$AB(P_1) = P_2$,

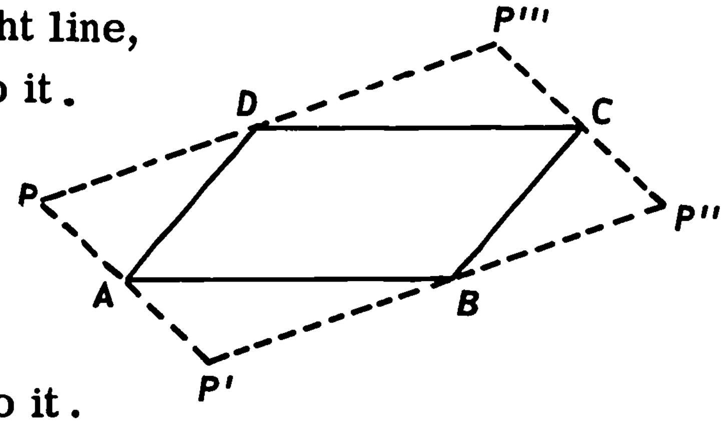
where $P_1P_2 = 2\,BA$ and parallel to it

But $CD = BA$; so $\overline{P_1P_2} \equiv \overline{PP_1}$.

Also $\overrightarrow{P_1P_2}$ points in the same direction as $\overrightarrow{P_1P}$.

Hence P_2 and P coincide,

i.e., $ABCD(P) = P$.



1. Distances are as indicated in the diagram.

$PP'' = p + p + (d - p) + (d - p)$
$= 2d.$

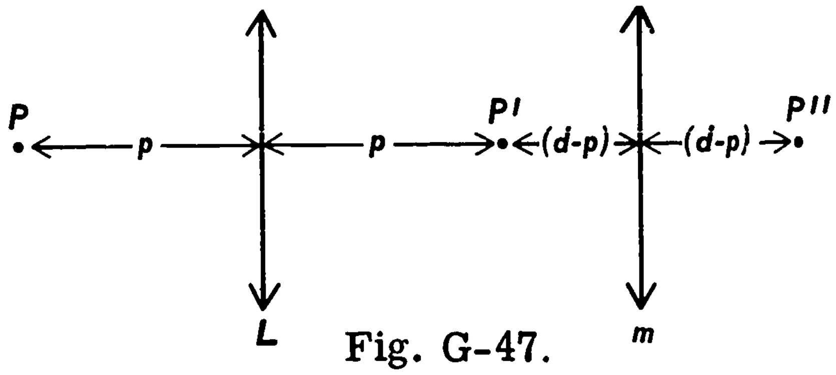


Fig. G-47.

2. Distances are as indicated in the diagram.

$PP_1'' = p + 2d - p$

$= 2d.$

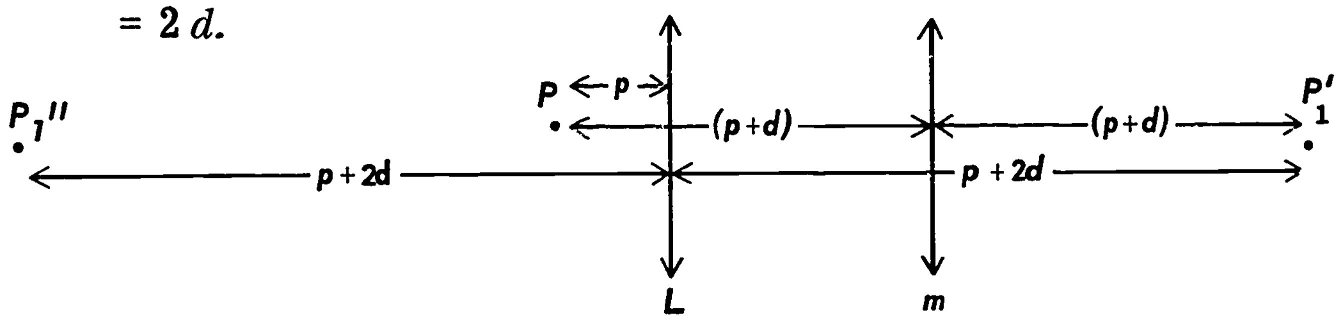


Fig. G-48.

3. Each point moves a distance $2d$ (if d is the separation of L and m). Hence a translation (see Problem 7 in Section 6-7).

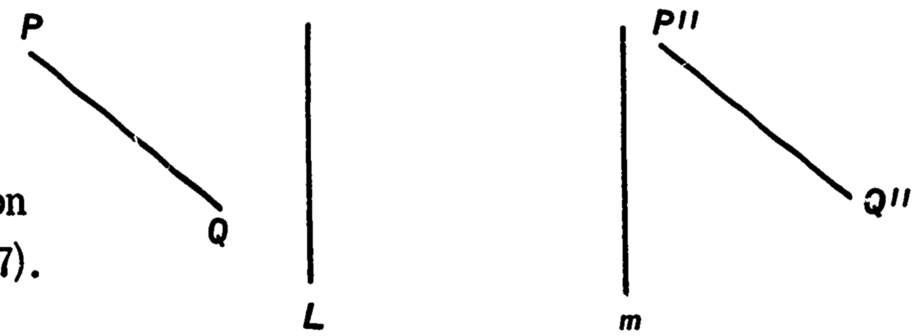


Fig. G-49.

4.

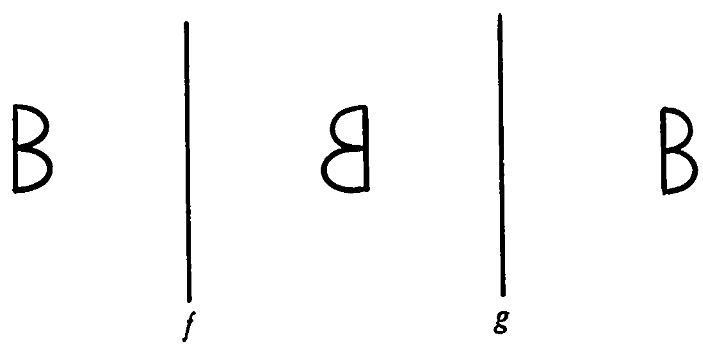


Fig. G-50.

5. S'', R'', Q'' and P'' give the required images. $SS'' = RR'' = QQ'' = PP'' = 6$ units.

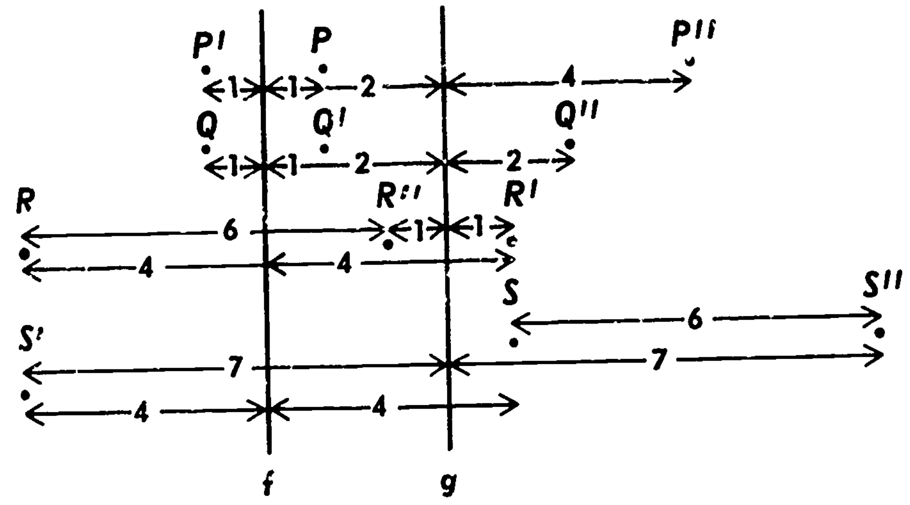


Fig. G-51.

6. S'', R'', Q'', P'' are the required images.
$SS'' = RR'' = QQ'' = PP'' = 6$ units.

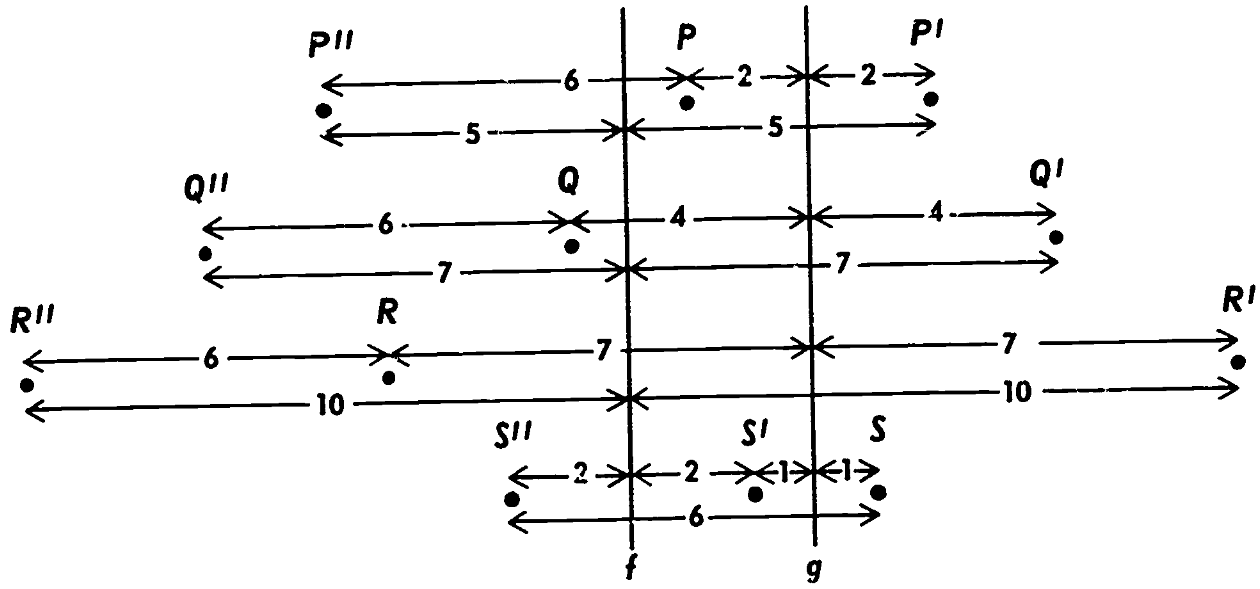


Fig. G-52.

7. Any two parallel lines perpendicular to **T** and whose distance of separation is half the length of **T** will be an acceptable solution.

6-11 MORE EXAMPLES OF THE USE OF MOTIONS.

General discussion. In this section we give four illustrations of the power of rigid motion ideas in solving geometrical problems. If you have had a traditional training in Euclidean geometry, you will find these examples especially impressive.

You should allow four class meetings if you wish to cover all four examples. You may select any subset of the four if you prefer, for the examples do not depend upon each other. You may wish to use only the first two examples. These are simpler than the last two.

Note that both of the first two examples have very simple congruent triangle solutions. What is important in these examples is not so much that rigid motions give the pupil shorter proofs (though they do do that) but rather that they give him an entirely new way of thinking about the problems. This new way of thinking about geometrical problems makes the solutions of many problems easier to find.

Note in Examples 3 and 4 the very powerful use made of the basic facts and rules given at the end of Section 6-10.

Answers to
PROBLEMS 6-11A

Student Text Pages 247-248

1. 60° (See the fact stated just before Problems 6-8A and proved in Problem 12 of Problems 6-8A).
2. $\overline{NC}$ goes to $\overline{AM}$ by a rotation through 90° about B. The isometric property of this rotation gives the result immediately.
3. Immediate by fact stated just before Problems 6-8A.

Answers to
PROBLEMS 6-11B

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1. The reflection in L carries $\overline{AB}$ onto $\overline{CB}$ and hence A onto C. By property (5) of rigid motions, we have the desired result.

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